Jurnal Pendidikan Matematika
https://doi.org/10.22342/jpm.v18i3.pp367-386

Analysis of Concept Construction and Student Errors on the Topic of Double Integral Based on APOS Theory

Yarman*, Fitrani Dwina, Dewi Murni, Yerizon, Kelly Angelly Hevardani

Department of Mathematics, Universitas Negeri Padang, Padang, Indonesia
*Email: yarman@fmipa.unp.ac.id

Abstract

Integral learning is particularly challenging for students, primarily due to misconceptions predominantly caused by students' lack of understanding about functions, limits, and derivatives. Therefore, this research aims to investigate students' thinking processes when solving double integral using Action, Process, Object, and Schema (APOS) theory, with a focus on past errors. In order to achieve the objective, a descriptive qualitative method was adopted. Data was collected from tests, interviews, and relevant documentation, and tested for validity using triangulation methods. The obtained results showed that high-ability students understood APOS stages in solving double integral. However, at the object stage, a lack of thoroughness in simplifying algebra led to misunderstandings. Medium-Ability Student (MS) was observed to successfully reach APOS stages when solving double integral using polar coordinates. Low-Ability Student (LS), on the other hand, showed inadequate understanding at the process stage, as evidenced by the failure to correctly draw the area and set integral boundaries. During the course of this investigation, process errors were found to be commonly associated with the calculations of double integral. In order to address these issues, Genetic Decomposition (GD) should be designed for other calculus topics, and error classification expanded to enhance the effectiveness of lectures.

Keywords: APOS Theory, Double Integral, Errors, Genetic Decomposition

How to Cite: Yarman., Dwina, F., Murni, D., Yerizon., & Hevardani, K. A. (2024). Analysis of concept construction and student errors on the topic of double integral based on APOS theory. *Jurnal Pendidikan Matematika, 18*(3), 367-386. https://doi.org/10.22342/jpm.v18i3.pp367-386

INTRODUCTION

A solid grasp of basic concepts is essential for studying advanced courses in calculus (Tall, 2014; Bressoud, 2021), such as integral, which is closely related to functions, series, limits, and derivatives (Dominguez et al., 2017; Borji et al., 2018; Ergene & Özdemir, 2022). This topic has been observed to play a crucial role in science and engineering (Hong et al., 2017; Martínez-Planell & Trigueros, 2021). As stated in previous research, understanding calculus requires knowledge of both derivatives and integral applications (Metaxas, 2007; Tall, 2011; Pepper et al., 2012). However, the majority of students tend to find integral difficult to comprehend (Ergene & Özdemir, 2020), due to the fact that the topic comprises the use of theorems, formulas, and interdisciplinary methods (Ergene & Özdemir, 2022). According to various previous investigations, students' comprehension of functions (Syarifuddin & Sari, 2021), limits (Bansilal et al., 2021), continuity (Perfekt, 2021), derivatives (Bangaru et al., 2021; Lam et al., 2021), integral (Fernandez & Mohammed, 2021), and rates of change (Avgerinos & Remoundou, 2021; Frank & Thompson, 2021) is often suboptimal. As a result, students tend to be less confident and motivated to study calculus (Bressoud et al., 2014).

Regarding the subject matter, Sealay (2008) found that students' understanding of integral was often limited to viewing the topic as an examination of the area under the curve or the inverse of the derivative, and many students struggled with the procedure for finding the area (Orton, 1983; Artigue, 1991). According to Ergene & Özdemir (2022) difficulties in studying this subject include

Received June 19, 2024; Revised August 9, 2024; Accepted August 15, 2024

understanding the relationship between Riemann sums and definite integral, determining integral limits, and solving integral. Subsequently, Kiat (2005) reported that students often faced challenges when solving integral of trigonometric functions and using concepts to determine areas. These observations were further emphasized by various previous investigations, where it was stated that the topic presented a significant challenge for both lecturers and students (Ferrini-Mundi & Graham, 1994; Rasslan & Tall, 2002; Ergene, 2014; Burgos et al., 2021). In an effort to improve the understanding of students, Mahir (2009) and Chapell & Kilpartrick (2003) recommended that conceptual learning should be focused on, rather than the adoption of a process-based approach.

Integral can be extended to multivariable functions with two or more independent variables. In both single and double integral functions of a variable, the area of integration is a closed interval in R and R^2, respectively. According to a previous exploration, the majority of students analyzed double integral over rectangular areas, non-rectangular areas, and polar coordinates (Martínez-Planell & Trigueros, 2021). The investigation further stated that a solid understanding of single-variable calculus was essential for learning multivariable calculus. This is because, without a good grasp of ordinary integral, students will find it challenging to solve double integral. Common mistakes made when solving this mathematical problem are often associated with drawing the intended area, the determination of intersection points and boundaries, application of properties and methods, as well as the performance of algebraic calculations (Orton, 1983).

The observation was further supported by Seah (2005), who stated that students struggled with drawing functions in polar coordinates, determining areas and boundaries, writing integral forms in polar coordinates, and performing algebraic calculations. According to Li et al. (2017), students often feel confused about identifying and using appropriate integral methods, emphasizing the essentiality of mastering different methods for solving mathematical problems. This difficulty, based on previous investigations, arises from a lack of understanding in drawing graphs for two and three-dimensional spaces, basic mathematical skills, and accuracy in algebraic calculations (Seah, 2005). Furthermore, it is important to state that poor performance in calculus has been observed to be frequently caused by errors in using symbols, notation, and variables (Tall, 1985; White & Mitchelmore, 1996), as well as when performing algebraic operations (Talley, 2009).

Orton (1983) classified errors into three types namely Arbitrary Errors (AE), which include ignoring the constraints specified in the problem, Structural Errors (SE), typically resulting from a failure to understand the principles of the solution, and Executive Errors (ExE), generally caused by incorrect manipulation. Meanwhile, Newman (1977) identified five types of errors in this regard, namely Reading Errors (RE), which occur when there is an inability to understand the meaning of each word, term, or symbol in the problem, Comprehension Errors (CE), caused by the inability to obtain the necessary information to solve the problem, Transformation Errors (TE), arising from an inability to understand the solution method, Process Skill Errors (PE), which occur as a result of incorrect use of concepts and calculation operations, and Encoding Errors (EnE), referring to the writing of incorrect conclusions. According to Li et al. (2017), repeated errors lead to frustration in learning mathematics, hence, an understanding of the mental construction of students in solving double integral is crucial as it can help teachers and lecturers address learning difficulties (Swan, 2001; Siyepu, 2015). A prominent solution to facilitate teachers' understanding of the mental construction of students includes the adoption of Action, Process, Object, and Schema (APOS) theory.

The theory has been observed to be instrumental in determining the formation and level of mathematical knowledge. It explains the understanding of mathematical concepts through the construction and use of mental structures (Arnon et al., 2014; García-Martínez & Parraguez, 2017; Syamsuri & Santosa, 2021). According to previous research, mental construction occurs through actions, processes, and objects, which are organized into a schema to solve mathematical problems (Dubinsky, 2013). An action is a procedural activity that relies on external information to carry out a procedure and apply a formula while a process includes using a procedure similar to an action but without needing external information. An object, on the other hand, represents a conceptual understanding where students can connect definitions, properties, and characteristics of certain materials to carry out a procedural activity. This collection of actions, processes, and objects forms a schema (Arnon et al., 2014).

According to Maharaj (2014), individuals cannot understand concepts when these mental constructions do not exist, hence, the importance of APOS theory. Marsitin (2017) stated that APOS theory could be applied to learning topics such as calculus, algebra, statistics, and discrete mathematics, among others. The theory has further been observed by Oktaç et al. (2019) to possess the capability to guide the role of teachers in aspects of learning, interactions, reflection activities, assignments, activities, and exercises. Following this theory, another framework predominantly used in mathematics education to understand and describe the cognitive processes associated with learning mathematical concepts is Genetic Decomposition (DG). Previous investigations reported the necessity of the implementation of GD, with a consideration on the fact that it serves as a model of the mental constructions needed in learning mathematical concepts (Arnon et al., 2014; Zwanch, 2019). As stated by Martínez-Planell et al. (2022), GD serves the role of an initial hypothesis, tested through interviews, and used as a basis for designing didactical activities.

APOS theory, on the other hand, has been used across different investigations to explore the thinking processes of students in various mathematical areas, including functions (Bansilal et al., 2017; Martínez-Planell & Trigueros, 2019; Şefik & Dost, 2020; Díaz-Berrios & Martínez-Planell, 2022), algebra (Harel, 2017), gradients (Nagle et al., 2019), limits (Baye et al., 2021), integral (Martínez-Planell & Trigueros, 2020; Borji & Martínez-Planell, 2023), matrices (Figueroa et al., 2017), and the principle of mathematical induction (García-Martínez & Parraguez, 2017). However, research on the thinking processes of students in solving double integral is limited. This observation was further supported by Martínez-Planell & Trigueros (2021) who stated that research on multivariable calculus was rarely conducted. Understanding double integral is crucial for calculating the area center, volume of a solid object, center of mass, moment of inertia, and curved surface area. Therefore, analyzing the mental structures of students and errors associated with solving double integral is essential. It is important to state that error analysis is particularly important within the present context as a comprehensive examination of errors can facilitate the understanding of lecturers about students' cognitive resources (Li et al., 2017). In order to achieve the research objectives, the following questions were addressed: 1) What kind of DG is used to investigate students' understanding when solving double integral?; 2) What are the results obtained from DG?; 3) What are the potential mistakes made when solving double integral?.

METHODS

This research included the participation of 67 fourth-semester students from Faculty of Mathematics and Natural Sciences Padang State University (FMIPA UNP) mathematics study program, who took the Multivariable Calculus course during the July – December 2023. The abilities of students were analyzed and categorized into three groups based on test results, including high (Value ≥ Mean + SD), medium (Mean – SD ≤ Value < Mean + SD), and low (Value < Mean – SD) groups.

Genetic Decomposition (GD) was designed to guide the double integral learning process (Figure 1) and interviews were conducted to determine the thought processes required for solving two mathematical questions. Accordingly, to determine the validity of the research items, the Pearson Product Moment formula was used and the results includes $r_{xy} = 0.45$ and $r_{xy} = 0.55$. Based on predefined standards, since $r_{xy} > r_{\text{tabel}}(5\%) = 0.244$, the research items were considered valid. Reliability test was also carried out in using the Cronbach Alpha formula and the obtained results showed that $r_{11} = 0.462$. This emphasized that the questions included medium-reliability criteria.

The mathematical questions used for the analysis include:

1. *Calculate the value of* $\iint_R (x+1)\,dA$ *where* R *is the area bounded by* $y = x$ *and* $y = 2 - x^2$.
2. *Determine* $\iint \sqrt{4 - x^2 - y^2}\,dA$ *with* R *where it is the quadrant I area of the circle* $x^2 + y^2 = 4$.

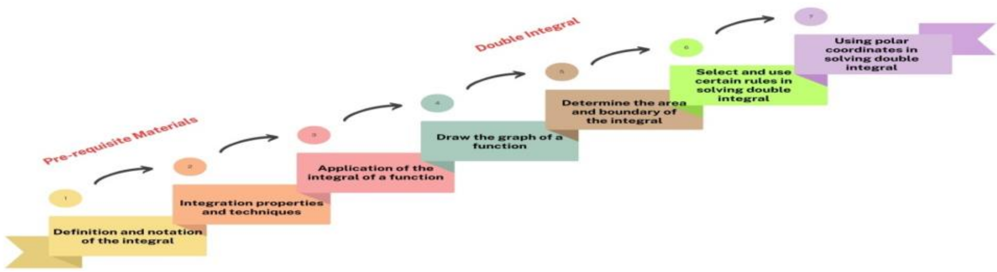


Figure 1. GD for the double integral topic

GD was designed based on the experiences of teachers who have taught the topic of double integral. Essentially, drawing graphs is the primary skill needed to solve double integral, due to the fact that integral area can only be analyzed successfully once the graph is accurately drawn. In this context, students can determine the points of intersection of the lines and the boundaries of integral. Additionally, recalling the characteristics of integral and integration methods is crucial, along with using polar coordinates to solve double integral. Another important skill in this aspect includes having precision in performing algebraic calculations.

Students were asked to solve two double-integral questions, and the answers were collected, evaluated, and classified as either correct, partially correct, or incorrect. An answer was categorized as correct with a score of 20 if it included drawing the area, determining the intersection points of the lines and the boundary integral, as well as solving double integral. If double integral was completed without drawing the area, the answer was classified as partially correct with a score of 10. All other answers were categorized as incorrect with a score of 0. Subsequently, students were grouped according to respective achievements at APOS stage using the indicators outlined in Table 1.

Table 1. Indicators of concept understanding based on APOS theory

APOS Stages	Indicators
Action	Students: 1) Identify what is known and ask the question. 2) Calculate multiple values of linear and quadratic functions.
Process	Students: 1) Determine the intersection point of the line. 2) Determine the coordinates of the vertex of the parabola.
Object	Students: 1) Draw the Cartesian coordinate axes. 2) Draw the *R* area. 3) Determine integral limit. 4) Select and use certain rules or procedures in solving double integral.
Schema	Students: 1) Relate the concept of integral to polar coordinates. 2) Select and use certain rules for polar coordinates correctly.

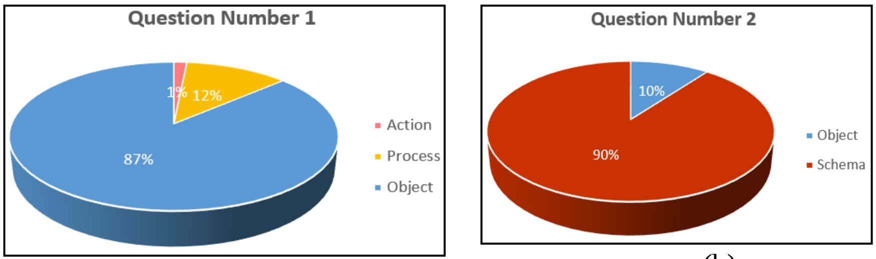
Common errors associated with solving double integral were analyzed using the classifications by Orton (1983) and Newman (1977), as shown in Table 2. Furthermore, interviews were conducted with 3 students of varying abilities to gather more information and understand respective thinking patterns.

Table 2. Classification of errors in completing double integral

Answers	Error Classification	
	Orton	Newman
Inability to draw graphs and determine double integral.	AE	RE
Failure to determine the point of intersection of lines and integral limit.	SE	CE TE
Failure to carry out algebraic calculation operations and write conclusions.	ExE	PE EnE

RESULTS AND DISCUSSION

The results obtained from solving the two double integral questions based on APOS theory are presented in Figure 2. For question number 1, 1% of students reached the action stage, 12% reached the process stage, and 87% reached the object stage. In its entirety, 80% made mistakes, while only 7% succeeded in determining the double integral correctly.



(a) (b)

Figure 2. Achievement of APOS stages in solving double-integral

As presented in Figure 2b, 10% of students reached action, process, and object stages, while others progressed to APOS stages. This shows that approximately 36% made mistakes, and 54% succeeded in determining double integral correctly. The errors made by students when solving the questions were also analyzed, as reported in Table 3.

Table 3. Percentage of errors in solving double integral

Question	Error Classification							
	Orton			Newman				
	AE	SE	ExE	RE	CE	TE	PE	EnE
1	1.49	38.80	52.24	1.49	1.49	37.31	52.24	-
2	-	8.96	37.31	-	-	8.96	37.31	-

After solving question number 1, 1.49% of students made AE or RE due to failure to draw the area and solve double integral. Approximately 38.80% made SE, consisting of 1.49% CE and 37.31% TE. Meanwhile, 52.24% made ExE or PE and were not able to carry out integral procedures correctly. Some examples of errors made by students when solving double integral are presented in Figure 3.

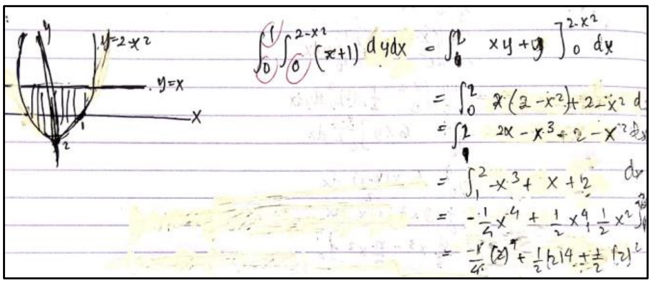


Figure 3. Errors on question number 1

As presented in Figure 3, students made mistakes when drawing the graph, and this led to the determination of an incorrect area. Ideally, an incorrect area leads to the determination of a wrong integral boundary, and this shows that integral would not be calculated to completion. Considering these errors, the ability to determine and apply integral methods becomes crucial for understanding the topic (Sofronas, 2011). In the present context, students failed to understand linear graphs and were unable to correctly determine the intersection points and integral boundaries.

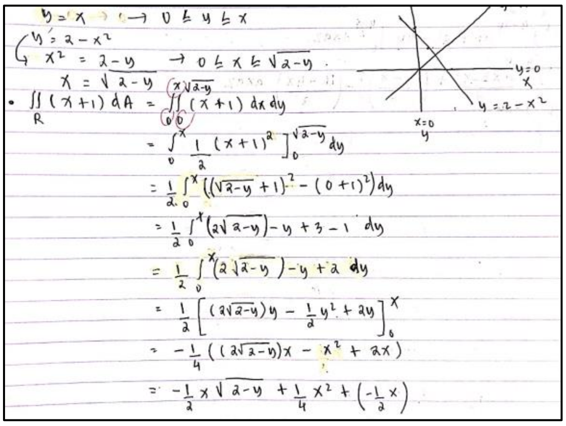


Figure 4. Errors on question number 1

Based on Figure 4, students made a mistake in drawing the $y = 2 - x^2$ graph and identifying the intersection point of the two graphs. The results were incorrect primarily because integral boundary included variables, meanwhile the ideal outermost integral boundary should be a number with no variables. As a result, some of students were unable to correctly draw the area and determine the intersection points as well as integral boundary. These errors are called CE, which according to Li et al. (2017), are classified as difficult problems since the concept is related to cognitive and mental abilities in recognizing functions as well as selecting and deciding appropriate integral methods. Furthermore, Orton (1983) stated that the errors could be categorized as SE.

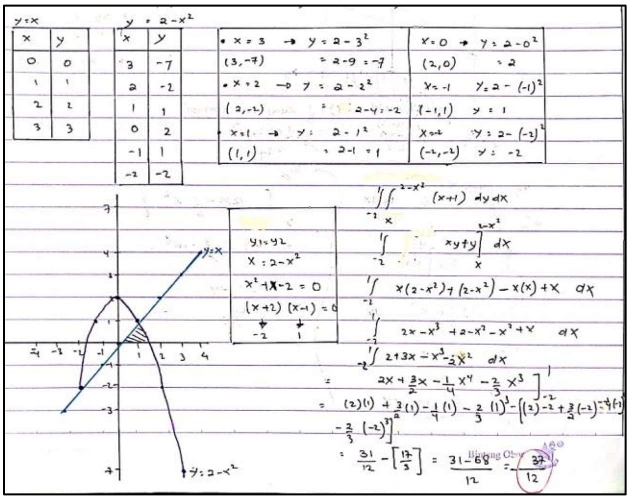


Figure 5. Errors on question number 1

Dissimilar to Figure 4 and Figure 5 presented the solution of a student who succeeded in drawing the graphs and correctly determined the intersection points of the lines. Several values of linear and quadratic functions were calculated to enhance the accuracy of the graphs. However, students made a mistake in determining the area and performing algebraic calculation operations.

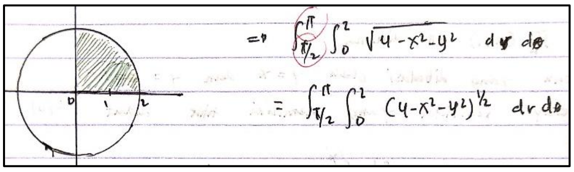


Figure 6. Errors on question number 2

To solve question number 2, students drew the area R consisting of the I quadrant area of the $x^2 + y^2 = 4$ circle, as shown in Figure 6. In this context, a strategy was selected to solve the question using polar coordinates but mistakes were made in determining integral boundary of $d\theta$. From the observation, it can be seen that the solution to the given question was not completed.

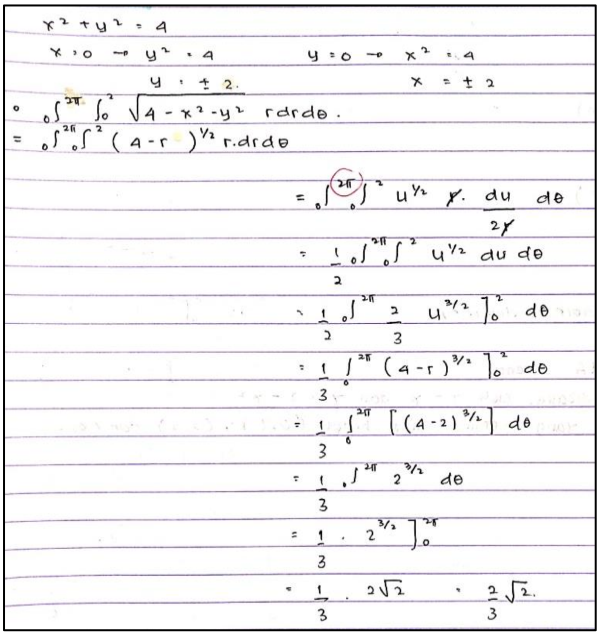


Figure 7. Errors on question number 2

As shown in Figure 7, the solution was provided for double integral using polar coordinates and mistakes were made in determining integral limit of $d\theta$, namely 0 to 2π. In this context, area R was not drawn due to incorrect integral results. These errors were categorized as SE or TE, where students did not draw the area R or determine integral limit correctly.

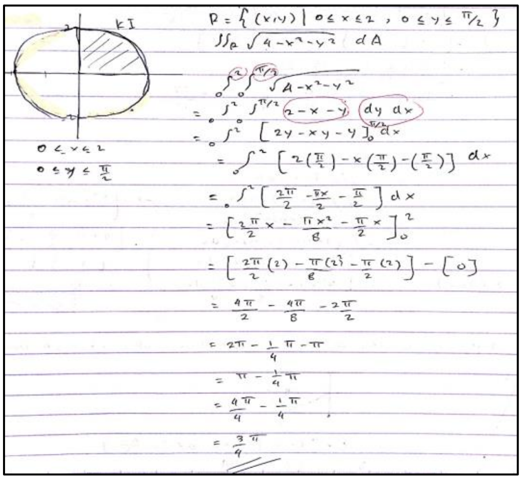


Figure 8. Errors on question number 2

Based on Figure 7 and Figure 8, observations were made that students succeeded in drawing the area R but made mistakes in determining integral limit, as reported in Figure 8. In the solution, students simplified $\sqrt{4-x^2-y^2}$ into $2-x-y$ and determined integral limit using angles. Considering these mistakes, double integral procedures could not be carried out using polar coordinates.

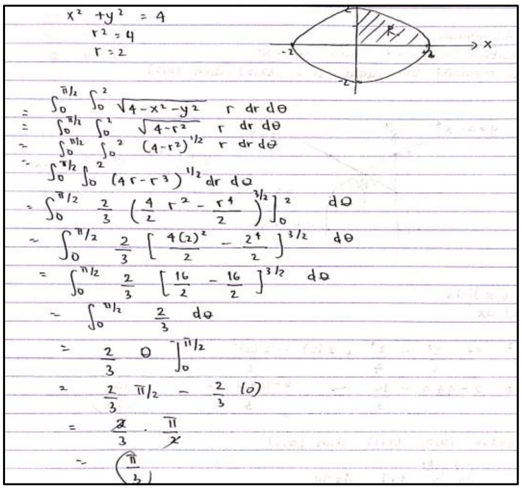


Figure 9. Errors on question number 2

In Figure 9, polar coordinates were preferably used to determine integral limit. Students were not careful in multiplying $(4-r^2)^{\frac{1}{2}}$ by r to obtain $(4r-r^3)^{\frac{1}{2}}$ and the result was integrated into $\frac{2}{3}\left(\frac{4}{2}r^2-\frac{r^4}{2}\right)^{\frac{3}{2}}$. In this context, students did not understand integration methods, meanwhile, the ability to determine and carry out these methods is very important in understanding integral (Sofronas, 2011). The errors observed in this regard are categorized as PE or ExE. To further examine the students' thought processes, some snippets of the answers obtained, which represent individual ability levels were presented and analyzed.

High-Ability Student (HS)

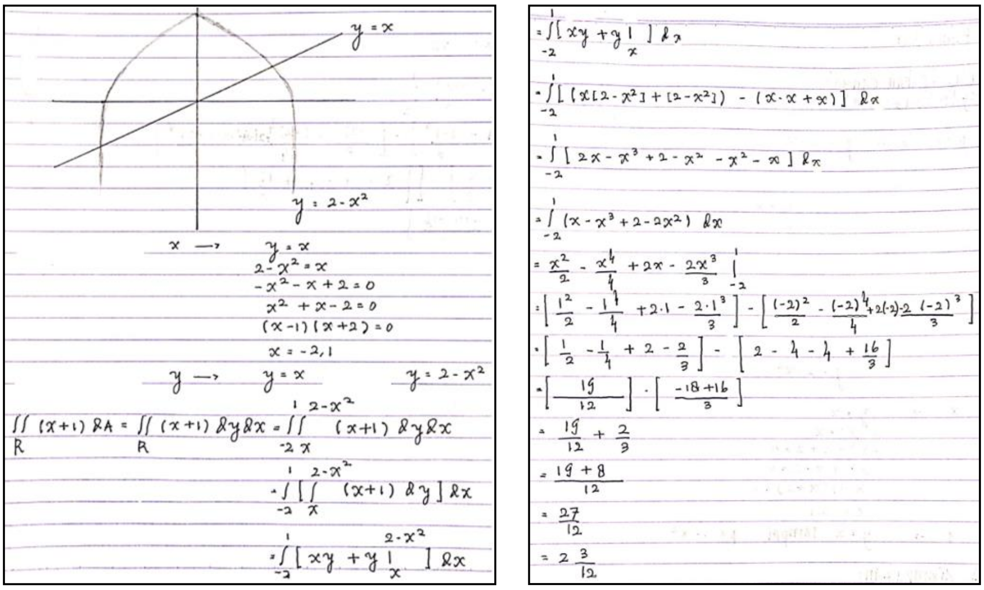


Figure 10. HS's answer to question number 1

Based on Figure 10, HS drew the area bounded by $y = x$ and $y = 2 - x^2$. Subsequently, the intersection point of the two graphs was determined by substituting $y = 2 - x^2$ for $y = x$ to obtain $x = -2$ and $x = 1$. It is also important to state that HS did not write the solution on the graph but on integral limit. To explore the understanding of solving double integral, the following interview was conducted.

Researcher	:	*What is your strategy for solving this problem?*
HS	:	*In this problem, we cannot directly calculate integral because integral limit is unknown. Therefore, we have to draw the graph first, namely the graph of* $y = x$ *and* $y = 2 - x^2$. *The graph of* $y = x$ *is in the form of a straight line, while the graph of* $y = 2 - x^2$ *is in the form of a parabola that opens downwards with the vertex of the parabola (0, 2) ma'am.*
Researcher	:	*So which area* R *is referred to in the problem?*
HS	:	*The area below graph* $y = 2 - x^2$ *and bounded by graph* $y = x$ *ma'am.*
Researcher	:	*OK, how do you determine integral limit?*
HS	:	*We substitute* $y = 2 - x^2$ *for* $y = x$ *to obtain* $x = -2$ *and* $x = 1$. *These two points are the intersection points of the two graphs and will also be integral limit of* dx. *Then for integral limit of* dy *it means from* $y = x$ *to* $y = 2 - x^2$ *ma'am.*
Researcher	:	*Why is that?*
HS	:	*If we look at the picture, the graph below area* R *is* $y = x$, *hence this is the lower limit of integral. Meanwhile, the graph above area* R *is* $y = 2 - x^2$ *hence this is the upper limit of integral, ma'am.*
Researcher	:	*What's the next step?*

HS	:	*Let us find integral with the limit we obtained earlier, ma'am. For the outer limit of integral, there cannot be any variable, which means they should be numbers. Meanwhile, the inner limit may have variables. Therefore, I looked for integral over dy first, then continued with integral over dx, and the result was $\frac{27}{12}$ ma'am.*
Researcher	:	*Do you think we can use $dxdy$?*
HS	:	*Yes, Ma'am, but we have to re-determine integral limit as before, ma'am.*
Researcher	:	*Will integral results be the same?*
HS	:	*It should be the same ma'am.*

Based on the interview, HS could identify what is known and ask about the problem, determine the intersection point of lines, draw the Cartesian coordinate axis and area R, determine integral limit, as well as select and use certain rules or procedures in solving double integral. Therefore, an inference can be made that the student possessed a good understanding of action, process, and object stages in solving the problem.

Medium-Ability Student (MS)

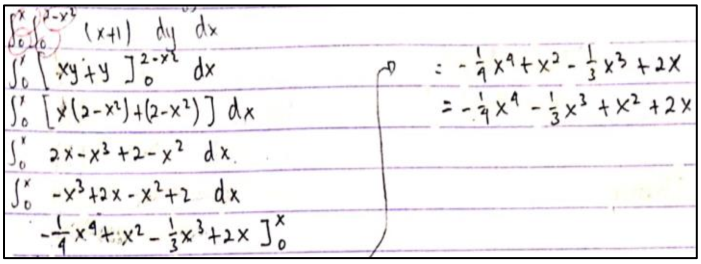


Figure 11. MS's answer to question number 1

Dissimilar to HS, according to Figure 11 MS did not draw the area R but directly used $y = x$ and $y = 2 - x^2$ for integral boundary. This caused integral results to be wrong and interviews were conducted as follows.

Researcher	:	*What is known and asked in question number 1?*
MS	:	*Given the area R which is bounded by $y = x$ and $y = 2 - x^2$. We are asked to calculate $\iint_R (x + 1)\, dA$, ma'am.*
Researcher	:	*What steps did you take to solve this problem?*
MS	:	*I integrate $x + 1$ over dy with the limit 0 to $2 - x^2$. Therefore I get $-x^3 + 2x - x^2 + 2$. Then the results are integrated again over dx with the lower limit being 0 and the upper limit being x. Here I get $-\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2 + 2x$ ma'am.*
Researcher	:	*Can you explain how you determine the limit of dy and dx?*

MS	:	*For limit dy it is obtained from equation $y = 2 - x^2$, while limit dx is obtained from equation $y = x$ ma'am. Because the lower limit is unknown, I just wrote it as 0.*
Researcher	:	*Is it permissible for the outer boundary, in this case, dx, to have variables?*
MS	:	*Hmm...maybe ma'am. Because the equation is known to have all the variables, ma'am.*
Researcher	:	*Do you think we need to draw both graphs?*
MS	:	*Maybe it is necessary ma'am (answers hesitantly).*
Researcher	:	*Then why did noy you draw the graph?*
MS	:	*Because integral limits are already known in the problem, I will not draw the graph again, ma'am.*

Based on the interview, MS could identify the solution and question. However, TE or SE was made in determining integral limit because of the failure to draw area R. In this context, MS understood the action stage in solving double integral.

Low-Ability Student (LS)

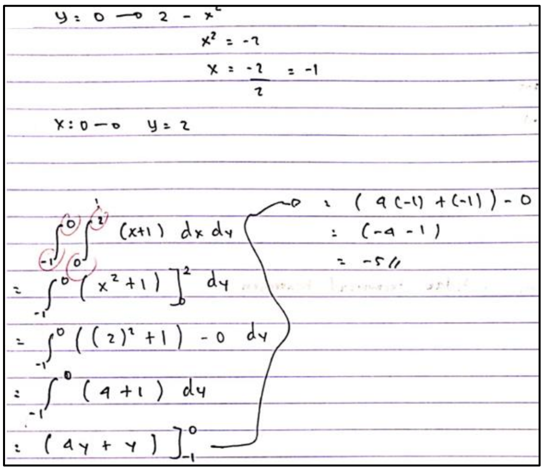


Figure 12. LS's answer to question number 1

Based on Figure 12, LS did not draw the area R but directly determined integral limit by substituting $x = 0$ and $y = 0$ into $y = 2 - x^2$. In the solution provided, a mistake was made where $x^2 = -2$ became $x = \frac{-2}{2} = -1$. The result was used for integral limit and interviews were conducted to explore the understanding of LS as follows.

Researcher	:	*What is known and asked in question number 1?*
LS	:	*We calculate $\iint_R (x + 1)\, dA$ where R is the area bounded by $y = x$ and $y = 2 - x^2$ ma'am.*

Researcher	:	*After you read the question, what steps did you take?*
LS	:	*I substitute* $y = 0$ *into equation* $y = 2 - x^2$ *to get* $x = -1$. *If I substitute* $x = 0$ *into equation* $y = 2 - x^2$ *we get* $y = 2$ *ma'am.*
Researcher	:	*Do you think we need to draw the graph first?*
LS	:	*It is necessary, but I am confused about how to make the graph, ma'am.*
Researcher	:	*How do you determine integral limit?*
LS	:	*From the results of this substitution ma'am (while pointing to the results of the previous substitution).* *In this case, the x limit starts from 0 to 2 and the y limit starts from* -1 *to 0 ma'am.*
Researcher	:	*So what is the next step?*
LS	:	*I integrate* $x + 1$ *over* dx, *and obtain* $(4 + 1)$. *Then it is integrated again over* dy *with a limit of* -1 *to 0 in obtaining the result* -5 *ma'am.*
Researcher	:	*Is it permissible for an area to be negative?*
LS	:	*Hmm... maybe not, ma'am (answered hesitantly).*

Based on the interview, LS could identify the question and solution. However, area R was not drawn but the line intersection point and integral boundary were determined correctly. In this context, the action and process steps in solving double integral were understood.

Table 4. Achievement of HS, MS, and LS on question number 1

Subject	APOS Stages/ Indicators							
	Action		Process		Object			
	1	2	1	2	1	2	3	4
HS	√	-	√	-	√	√	√	√
MS	√	-	-	-	-	-	-	-
LS	√	-	√	-	-	-	-	-

From the data presented in Table 4, MS and LS did not achieve all indicators at action, process, and object stages. This was due to the fact that the students failed to draw the area R, as a result of the wrong integral boundary obtained. Among the three categories, only HS was able to complete double integral. Based on the interview for question number 2, LS could draw the Cartesian coordinate axes and select polar coordinates. However, errors were made when drawing the area R and determining integral boundary. LS also did not differentiate the derivative and integral solving processes due to TE and SE. It is also important to comprehend that the student's understanding of basic algebra was weak.

Table 5. Achievement of HS, MS, and LS on question number 2

Subject	APOS Stages/Indicators									
	Action		Process		Object				Schema	
	1	2	1	2	1	2	3	4	1	2
HS	√	-	-	-	√	√	√	-	√	√
MS	√	-	-	-	√	√	√	√	√	√
LS	√	-	√	-	√	√	√	√	√	-

In Table 5, both HS and LS did not achieve all indicators at APOS stages. HS was not careful in carrying out algebraic calculations, while LS selected not to complete integral due to confusion about the next steps. This is in accordance with the investigation by Borji & Font (2019), who stated that students often struggled with integration procedure. Only MS succeeded in solving double integral using polar coordinates. It is important to state that the second indicator at the action stage as well as the first and second indicators at the process stage were optional. The obtained results showed that HS and MS had similar thought process as the designed GD.

In solving double integral, PE or ExE were frequently observed. These errors have been previously identified by Li et al. (2017) as follows, (1) conceptual, including issues with the use of symbols, introduction of standard functions, integral properties, and methods, (2) procedural, such as confusion between derivative and integral processes, and (3) technical errors, including a lack of mathematical ability and accuracy. Accordingly, Kiat (2005) found that students often struggled to determine the area of the curve intersecting with the axis, this was categorized as a conceptual error. The research also emphasized that constant terms were often omitted when solving indefinite integral, leading to confusion between derivative and integral processes.

Previous research found that students commonly made errors with the chain rule, absolute value, partial derivatives, integral, and exponents, as well as determining final answers and performing calculation operations (Li et al., 2017). This information is capable of helping lecturers develop suitable learning strategies to enhance lecture effectiveness (Baye et al., 2021). A strategy that can be adopted in this regard is the in-class voting method. The method has been proven to address the stated errors effectively and create an active learning environment for discussing mathematical issues (Cline et al., 2013). It has also been found to significantly improve the learning outcomes of students (Miller et al., 2006; Zullo et al., 2011).

According to Talley (2009) students adopt different thinking patterns when solving problems. This led Wibawa et al. (2017) to investigate the thinking structures required for solving definite integral. According to the research, it is essential that students understand the topic of definite integral in terms of areas related to curves and limits, the representation of functions in geometry (Cartesian coordinates), function rules, intersection, and vertex points, as well as the positive value area as the difference between upper and lower curves. Other important areas to understand include the definite integral as related to Riemann sums, function rules, limits, and integration methods. As stated by Misu et al. (2019), both male and female students can use metacognitive knowledge and visual thinking to understand the concept of the indefinite integral when classifying and summarizing categories. However, only male students were observed to effectively use these metacognitive skills to understand integral concepts.

Sholihah & Maryono (2020) classified visual thinking abilities in solving integral into three levels, namely (1) semi-locall, where students understand integral algebraically, (2) local, where students grasp geometry and can represent problems graphically, and (3) global visual levels, where students do not understand integral both algebraically and geometrically. Still considering the subject matter, Borji & Font (2019) combined APOS and OSA theories to analyze students' understanding of partial integral, as these theories complement each other in conceptualizing mathematical objects (Bikner-Ahsbahs & Prediger, 2014; Font et al., 2016; Borji et al., 2018). Based on observations, many students struggle to make correct choices for integration methods (Mateus, 2016). For instance, at the intra-level, 56% of students had difficulty understanding prerequisite materials, such as functions, derivatives, and basic integral. This is consistent with the observation made by Mahir (2009), who found that the inability of students to master derivatives, a prerequisite material, led to the inability to complete integral.

In accordance with this, Brijlall & Ndlazi (2019) reported that students' understanding of integral in engineering was primarily at the action stage, with some signs of process conceptualization in predicting integral methods used. During the investigation, it was observed that the majority of students' understanding was procedural, focusing more on integral concept than on derivatives. Additionally, students could only relate definite integral to areas when provided with external stimulation. Various previous research stated by McGee & Martínez-Planell (2014) emphasized that a proper understanding of functions and derivatives was crucial for engineering students studying integral.

CONCLUSION

In conclusion, based on the GD designed, an inference was made that a proper understanding of functions and derivatives is a prerequisite for solving double integral. Furthermore, it was observed that a key skill required by students to solve integral-related problems is the ability to properly draw graphs. This was primarily because once students had successfully drawn the graph, the analysis of the relevant integral area, determination of the intersection points of the lines, and the establishment of integral boundaries could be carried out effectively.

The results obtained from the analysis showed that HS understood APOS stages in solving double integral. However, a lack of thoroughness in simplifying algebra led to misunderstandings at the object stage. The majority of MS generally reached only action stage in solving double integral, while some succeeded in reaching APOS stages when using polar coordinates. In accordance, LS was observed to struggle at the process stage due to the failure to draw the area first, resulting in incorrect integral boundaries. It is also important to state that weaknesses in basic algebra were found to affect students' understanding and achievement in calculus. During the investigation, students often made process or executive errors, and based on these issues, future research was recommended to aim toward designing DG models for other calculus topics and expand the classification of errors.

ACKNOWLEDGMENTS

The authors would like to thank Lembaga Penelitian dan Pengabdian Masyarakat Universitas Negeri Padang for funding this work with contract number 1670/UN35.15/LT/2024.

REFERENCES

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*. New York: Springer. http://dx.doi.org/10.1007/978-1-4614-7966-6

Artigue, M. (1991). Analysis in Advanced Mathematical Thinking.

Avgerinos, E., & Remoundou, D. (2021). The Language of Rate of Change in Mathematics. *European Journal of Investigation in Health, Psychology and Education*, *11*, 1599–1609. https://doi.org/10.3390/ejihpe11040113

Bangaru, S. P., Michel, J., Mu, K., Bernstein, G., Li, T. M., & Ragan-Kelley, J. (2021). Systematically Differentiating Parametric Discontinuities. *ACM Transactions on Graphics*, *40*(4), 1–18. https://doi.org/10.1145/3450626.3459775

Bansilal, S., & Mkhwanazi, T. W. (2021). Pre-service Student Teachers' Conceptions of the Notion of Limit. *International Journal of Mathematical Education in Science and Technology*, 1–19. https://doi.org/10.1080/0020739X.2020.1864488

Bansilal, S., Brijlall, D., & Trigueros, M. (2017). An APOS Study on Pre-service Teachers' Understanding of Injections and Surjections. *Journal of Mathematical Behavior*, *48*, 22–37. https://doi.org/10.1016/j.jmathb.2017.08.002

Baye, M. G., Ayele, M. A., & Wondimuneh, T. E. (2021). Implementing GeoGebra Integrated with Multi-teaching Approaches Guided by the APOS Theory to Enhance Students' Conceptual Understanding of Limit in Ethiopian Universities. *Heliyon*, *7*, 1–13. https://doi.org/10.1016/j.heliyon.2021.e07012

Bikner-Ahsbahs, A., & Prediger, S. (2014). *Networking of Theories as a Research Practice in Mathematics Education*. Switzerland: Springer. https://doi.org/10.1007/978-3-319-05389-9

Borji, V., & Font, V. (2019). Exploring Students' Understanding of Integration by Parts: A Combined Use of APOS and OSA. *EURASIA: Journal of Mathematics, Science and Technology Education, 15*(7), 1–13. https://doi.org/10.29333/ejmste/106166

Borji, V., & Martínez-Planell, R. (2023). On Students' Understanding of Volumes of Solids of Revolution: An APOS Analysis. *Journal of Mathematical Behavior, 70*, 1–20. https://doi.org/10.1016/j.jmathb.2022.101027

Borji, V., Alamolhodaei, H., & Radmehr, F. (2018). Application of the APOS-ACE Theory to Improve Students' Graphical Understanding of Derivative. *EURASIA Journal of Mathematics, Science and Technology Education, 14*(7), 2947–2967. https://doi.org/10.29333/ejmste/91451

Bressoud, D. M. (2021). The Strange Role of Calculus in the United States. *ZDM–Mathematics Education, 53*(3), 521–533. https://doi.org/10.1007/s11858-020-01188-0

Bressoud, D., Burn, H., Hsu, E., Mesa, W., Rasmussen, C., & White, N. (2014). *Successful Calculus Programs: Two-year Colleges to Research Universities*. USA: NCTM.

Brijlall, D., & Ndlazi, N. J. (2019). Analyzing Engineering Students' Understanding of Integration to Propose a Genetic Decomposition. *Journal of Mathematical Behavior, 55*, 1–12. https://doi.org/10.1016/j.jmathb.2019.01.006

Burgos, M., Bueno, S., Godino, J. D., & Pérez, O. (2021). Onto-semiotic Complexity of the Definite Integral: Implications for Teaching and Learning Calculus. *REDIMAT – Journal of Research in Mathematics Education*, *10*(1), 4–40. https://doi.org/10.17583/redimat.2021.6778

Chapell, K. K., & Kilpartrick, K. (2003). Effects of Concept-based Instruction on Students' Conceptual Understanding and Procedural Knowledge of Calculus. *PRIMUS*, *13*(1), 17–37. https://doi.org/10.1080/10511970308984043

Cline, K., Parker, M., Zullo, H., & Stewart, A. (2013). Addressing Common Student Errors with Classroom Voting in Multivariable Calculus. *PRIMUS*, *23*(1), 60–75. https://doi.org/10.1080/10511970.2012.697098

Díaz-Berrios, T., & Martínez-Planell, R. (2022). High School Student Understanding of Exponential and Logarithmic Functions. *Journal of Mathematical Behavior, 66*, 1–20. https://doi.org/10.1016/j.jmathb.2022.100953

Dominguez, A., Barniol, P., & Zavala, G. (2017). Test of Understanding Graphs in Calculus: Test of Students' Interpretation of Calculus Graphs. *EURASIA Journal of Mathematics, Science and Technology Education*, *13*(10), 6507–6531. https://doi.org/10.12973/ejmste/78085

Dubinsky, E. (2013). Using a Theory of Learning in College Mathematics Courses. *MSOR Connections, 1*(2), 10–15.

Ergene, Ö. & Özdemir, A. Ş. (2020). Investigating Pre-service Elementary Mathematics Teachers' Perception of Integral. *Journal of Educational Sciences, 51*(51), 155–176. https://doi.org/10.15285/maruaebd.622149

Ergene, Ö. (2014). *Investigation of Personal Relationship in Integral Volume Problems Solving Process within Communities of Practices*. Dissertation. Marmara University.

Ergene, Ö., & Özdemir, A. Ş. (2022). Understanding the Definite Integral with the Help of Riemann Sums. *Participatory Educational Research, 9*(3), 445–465. https://doi.org/10.17275/per.22.75.9.3

Fernandez, A., & Mohammed, P. (2021). Hermite-Hadamard Inequalities in Fractional Calculus Defined Using Mittag-Leffler Kernels. *Mathematical Methods in the Applied Sciences, 44*(10), 8414–8431. https://doi.org/10.1002/mma.6188

Ferrini-Mundi, J., & Graham, K. (1994). Research in Calculus Learning: Understanding of Limits, Derivatives and Integrals. In J. J. Kaput & E. Dubinsky (eds.), *Research Issues in Undergraduate Mathematics Learning*, (pp.31–45). Washington DC: MAA.

Figueroa, A. P., Possani, E., & Trigueros, M. (2017). Matrix Multiplication and Transformations: An APOS Approach. *Journal of Mathematical Behavior*, 1–15. https://doi.org/10.1016/j.jmathb.2017.11.002

Font, V., Trigueros, M., Badillo, E., & Rubio, N. (2016). Mathematical Objects through the Lens of Two Different Theoretical Perspectives: APOS and OSA. *Educational Studies in Mathematics, 91*(1), 107–122. https://doi.org/10.1007/s10649-015-9639-6

Frank, K., & Thompson, P. W. (2021). School Students' Preparation for Calculus in the United States. *ZDM–Mathematics Education, 53*(3), 549–562. https://doi.org/10.1007/s11858-021-01231-8

García-Martínez, I., & Parraguez, M. (2017). The Basis Step in the Construction of the Principle of Mathematical Induction Based on APOS Theory. *Journal of Mathematical Behavior*, *46*, 128–143. https://doi.org/10.1016/j.jmathb.2017.04.001

Harel, G. (2017). The Learning and Teaching of Linear Algebra: Observations and Generalizations. *The Journal of Mathematical Behavior, 46*, 69–95. https://doi.org/10.1016/j.jmathb.2017.02.007

Hong, D. S., Choi, K. M., Hwang, J., & Runnalls, C. (2017). Integral Students' Experiences: Measuring Instructional Quality and Instructors' Challenges in Calculus 1 Lessons. *International Journal of Research in Education and Science, 3*(2), 424–437. https://doi.org/10.21890/ijres.327901

Kiat, S. E. (2005). Analysis of Students' Difficulties in Solving Integration Problems. *The Mathematics Educator, 9*(1), 39–59.

Lam, T. T., Guan, T. E., & Luen, T. C. (2021). Fallacies About the Derivative of the Trigonometric Sine Function. *The Mathematician Educator, 2*(1), 1–10.

Li, V. L., Julaihi, N. H., & Eng, T. H. (2017). Misconceptions and Errors in Learning Integral Calculus. *Asian Journal of University Education, 13*(1), 17–39. https://ir.uitm.edu.my/id/eprint/21914

Maharaj, A. (2014). An APOS Analysis of Natural Science Students' Understanding of Integration. *REDIMAT – Journal of Research in Mathematics Education, 3*(1), 54–73. https://doi.org/10.4471/redimat.2014.40

Mahir N. (2009). Conceptual and Procedural Performance of Undergraduate Students in Integration. *International Journal of Mathematical Education in Science and Technology, 40*(2), 201–211. https://doi.org/10.1080/00207390802213591

Marsitin, R. (2017). Koneksi Matematis dan Berpikir Kreatif dalam Pembelajaran Matematika dengan Teori APOS [Mathematical Connection and Creative Thinking in Mathematics Learning with APOS Theory]. *Al-Khwarizmi: Jurnal Pendidikan Matematika dan Ilmu Pengetahuan Alam*, *5*(1), 87–100. https://doi.org/10.24256/jpmipa.v5i1.268

Martínez-Planell, R., & Trigueros, M. (2019). Using Cycles of Research in APOS: The Case of Functions of Two Variables. *Journal of Mathematical Behavior, 55*, 1–22. https://doi.org/10.1016/j.jmathb.2019.01.003

Martínez-Planell, R., & Trigueros, M. (2020). Students' Understanding of Riemann Sums for Integrals of Functions of Two Variables. *Journal of Mathematical Behavior, 59*, 1–26. https://doi.org/10.1016/j.jmathb.2020.100791

Martínez-Planell, R., & Trigueros, M. (2021). Multivariable Calculus Results in Different Countries. *ZDM – Mathematics Education, 53*, 695–707. https://doi.org/10.1007/s11858-021-01233-6

Martínez-Planell, R., Trigueros, M., & Borji, V. (2022). The Relation Between Riemann Sums and Double Integrals: Results of a Second Research Cycle. *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education, 3*, 179–186. http://hdl.handle.net/10045/126627

Mateus, E. (2016). Teaching Analysis to Process Integration Method Instruction by Parties. *Bolema: Boletim de Educação Matemática, 30*(55), 559–585. https://doi.org/10.1590/1980-4415v30n55a13

McGee, D., & Martínez-Planell, R. (2014). A Study of Semiotic Registers in the Development of the Definite Integral of Functions of Two and Three Variables. *International Journal of Science and Mathematics Education, 12*(4), 883–916. https://doi.org/10.1007/s10763-013-9437-5

Metaxas, N. (2007). Difficulties on Understanding the Indefinite Integral. In Woo, J. H., Lew, H. C., Park, K. S., Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, 3*, 265–272.

Miller, R. L., Santana-Vega, E., & Terrell, M. S. (2006). Can Good Questions and Peer Discussion Improve Calculus Instruction? *PRIMUS, 16*(3), 193–203. https://doi.org/10.1080/10511970608984146

Misu, L., Budayasa, I. K., Lukito, A., Hasnawati., & Rahim, U. (2019). Profile of Metacognition of Mathematics Education Students in Understanding the Concept of Integral in Category Classifying and Summarizing. *International Journal of Instruction, 12*(3), 481–496. https://doi.org/10.29333/iji.2019.12329a

Nagle, C., Martínez-Planell, R., & Moore-Russo, D. (2019). Using APOS Theory as a Framework for Considering Slope Understanding. *Journal of Mathematical Behavior, 54*, 1–14. https://doi.org/10.1016/j.jmathb.2018.12.003

Newman, M. A. (1977). An Analysis of Sixth-grade Pupils' Error on Written Mathematical Tasks. *Victorian Institute for Educational Research Bulletin, 39*, 31–43. https://cir.nii.ac.jp/crid/1573105976160816128

Oktaç, A., Trigueros, M., & Romo, A. (2019). APOS Theory: Connecting Research and Teaching. *For the Learning of Mathematics, 39*(1), 33–37. https://www.jstor.org/stable/26742010

Orton, A. (1983). Students' Understanding of Integration. *Educational Studies in Mathematics, 14*(1), 1–18. https://doi.org/10.1007/BF00704699

Pepper, R., Stephanie, V. C., Steven J. P., & Katherine K. P. (2012). Observations on Student Difficulties with Mathematics in Upper-division Electricity and Magnetism. *Physical Review Special Topics – Physics Education Research, 8*(1), 1–15. https://doi.org/10.1103/PhysRevSTPER.8.010111

Perfekt, K. M. (2021). Plasmonic Eigenvalue Problem for Corners: Limiting Absorption Principle and Absolute Continuity in the Essential Spectrum. *Journal de Mathématiques Pures et Appliquées, 145*, 130–162. https://doi.org/10.1016/j.matpur.2020.07.001

Rasslan, S., & Tall, D. (2002). Definitions and Images for the Definite Integral Concept. *Proceedings of the 26th PME, 4*, 89–96.

Seah, E. K. (2005). Analysis of Students' Difficulties in Solving Integration Problems. *The Mathematics Educator, 9*(1), 39–59.

Sealey, V. (2008). *Calculus Students' Assimilation of the Riemann Integral into a Previously Established Limit Structure*. Dissertation. Arizona State University.

Şefik, Ö., & Dost, Ş. (2020). The Analysis of the Understanding of the Three-dimensional (Euclidian) Space and the Two-variable Function Concept by University Students. *Journal of Mathematical Behavior, 57,* 1–19. https://doi.org/10.1016/j.jmathb.2019.03.004

Sholihah, U., & Maryono. (2020). Students' Visual Thinking Ability in Solving the Integral Problem. *Journal of Research and Advances in Mathematics Education, 5*(2), 175–186. https://doi.org/10.23917/jramathedu.v5i2.10286

Siyepu, S. W. (2015). Analysis of Errors in Derivatives of Trigonometric Functions. *International Journal of STEM Education, 2*(16), 1–16. https://doi.org/10.1186/s40594-015-0029-5

Sofronas, K. S. (2011). What does it Mean for a Student to Understand the First-year Calculus? Perspectives of 24 Experts. *The Journal of Mathematical Behavior, 30*(2), 131–148. https://doi.org/10.1016/j.jmathb.2011.02.001

Swan, M. (2001). Dealing with Misconceptions in Mathematics. In Gates, P. (Ed.), *Issues in Mathematics Teaching*, (pp. 147–165). London: Routledge Falmer.

Syamsuri., & Santosa, C. (2021). Thinking Structure of Students' Understanding of Probability Concept in Term of APOS Theory. *MaPan: Jurnal Matematika dan Pembelajaran, 9*(1), 119–135. https://doi.org/10.24252/mapan.2021v9n1a8

Syarifuddin, A., & Sari, A. F. (2021). Misconceptions of Prospective Mathematics Teacher on Graphing Function. *Journal of Physics: Conference Series, 1869*(1), 1–6. https://doi.org/10.1088/1742-6596/1869/1/012115

Tall, D. (1985). Understanding the Calculus. *Mathematics Teaching, 110*, 49–53.

Tall, D. (2011). Looking for the Bigger Picture. *For the Learning of Mathematics*, *31*(2), 17–18.

Tall, D. (2014). Making Sense of Mathematical Reasoning and Proof. In Fried, M., Dreyfus, T. (Ed)., *Mathematics & Mathematics Education: Searching for Common Ground*, (pp. 223–235). New York: Springer. https://doi.org/10.1007/978-94-007-7473-5_13

Talley, J. R. (2009). *Calculus Instructors' Responses to Prior Knowledge Errors*. Dissertation. University of Oklahoma, United States.

White, P., & Mitchelmore, M. (1996). Conceptual Knowledge in Introductory Calculus. *Journal for Research in Mathematics Education, 27*(1), 79–95. https://doi.org/10.5951/jresematheduc.27.1.0079

Wibawa, K. A., Nusantara, T., Subanji., & Parta, I. N. (2017). Fragmentation of Thinking Structure's Students to Solving the Problem of Application Definite Integral in Area. *International Education Studies*, *10*(5), 48–60. https://doi.org/10.5539/ies.v10n5p48

Zullo, H., Cline, K., Parker, M., Buckmire, R., George, J., Gurski, K., Larsen, J., Mellor, B., Oberweiser, J., Peterson, D., Spindler, R., Stewart, A., & Storm, C. (2011). *Student Surveys: What do They Think?* Mathematical Association of America, 29–34. https://doi.org/10.1017/CBO9781614443018.006

Zwanch, K. (2019). A Preliminary Genetic Decomposition of Probabilistic Independence. *The Mathematics Educator*, *28*(1), 3–26. https://files.eric.ed.gov/fulltext/EJ1225414.pdf