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Exploring Mathematical Proofs and Solutions in Higher Education: A Case Study on Real Analysis

Dodi Isran[1,*], Agus Susanta[2], Dewi Rahimah[3], Fatrima Santri Syafri[4]

[1]STIES Nahdlatul Ulama Bengkulu, Indonesia
[2,3]Magister mathematics Education Study Program, Bengkulu University, Bengkulu, Indonesia
[4]Mathematics Education Study Program, UIN Fatmawati Sukarno Bengkulu, Indonesia
[*]Email: dodi.isran@gmail.com

Abstract

This exploratory study investigates the teaching and learning of mathematical proofs, specifically focusing on real nalysis proofs, in mathematics education programs at the higher education level in Bengkulu Province. Data were collected through in-depth interviews with lecturers and students, classroom observations, and document analysis of students' assignments. The findings reveal that lecturers employ various teaching strategies, such as active learning, step-by-step explanations, and the use of technology to aid understanding. However, students face significant challenges, including difficulties with abstract thinking, gaps in foundational knowledge, and the complex language of mathematical proofs. Despite these challenges, students reported improvements in their logical reasoning, problem-solving, and self-confidence as they were engaged in the construction mathematical proofs. Classroom observations confirmed that collaborative learning was effective in promoting understanding. Document analysis of students' assignments indicated a range of proficiency levels, with some students struggling to produce clear and logical proofs. The study highlights the importance of mathematical proofs in developing critical thinking skills and analytical abilities. It suggests that more interactive, student-centered teaching methods are necessary to address the challenges students face and improve learning outcomes. These findings provide valuable insights for enhancing teaching practices and supporting students' mastery of mathematical proofs.

Keywords: Mathematical Proof, Higher Education, Teaching Strategies, Critical Thinking, Student Challenges

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INTRODUCTION

Higher education plays a central role in developing individuals' intellectuality, especially in the field of mathematics. As a subject that occupies a special position in higher education, mathematics is not only a tool in various scientific disciplines, but also a vehicle for training students' critical and analytical thinking skills. Therefore, mathematical proof is one of the most important aspects of mathematics learning in higher education. In the learning of mathematical proofs, students are required to formulate and present strong logical arguments to validate the truth of mathematical statements, which ultimately sharpens their critical thinking skills. Mathematics acts not only as a tool in other scientific disciplines, but also as a vehicle for honing critical thinking skills and strong analytical skills (Hardy & Snow, 1992; Hasan, 2020; Jessup et al., 2021; Nonami et al., 2019; Polya, 2004). In addition, mathematical proofs also provide a solid foundation for a deep understanding of mathematical concepts, which is very useful in advanced courses and in contributing to academic research. Thus, mathematical proofs in higher education are not only an integral part of mathematics curriculum, but also an important foundation for developing in forming students' intellectual abilities.

Mathematical proof is a logical process used to verify the validity of mathematical statements. More than simply determining truth, mathematical proofs play an important role in developing the

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understanding of mathematical structures and in investigating new concepts (Hardy & Snow, 1992; Macwhinney, 2008; Nonami et al., 2019) . This statement indicates that mathematical proof is a process used to logically confirm the truth of a mathematical statement. Mathematical proof is not only limited to determining whether a statement is true or false. It also has a broader role. Mathematical proofs also contribute to the development of understanding of the overall structure of mathematics and assist in the investigation of new mathematical concepts. In other words, mathematical proofs are not just about "yes or no," but also about a deeper exploration and understanding of mathematics.

Data highlighting the importance of mathematical proofs in higher education comes from multiple sources. Alexanderson and Polya (1979) and Hardy and Snow (1992) emphasize the role of mathematics in developing critical thinking and analytical skills. Macwhinney (2008) and Nonami et al. (2019) highlight the broader role of proofs in understanding mathematical structures and discovering new theorems. B. and Bell (1937) and Velleman (2019) argue that mathematical proofs are essential for fostering critical thinking, which is necessary in scientific research and professional careers. Krantz (2017) and Strogatz (2013) discuss the far-reaching applications of mathematical proofs in fields such as physics, economics, and computer science. Čížková and Čížek (2012), along with Rota (1997) and Trefethen and Bau (2022), note that proof skills are also valuable in careers related to technology, data science, engineering, and finance. This study examines the role and importance of mathematical proofs in higher education. These sources suggest that mathematical proof is integral not only to the development of a student’s intellectual abilities but also to advancing research and solving complex problems in various fields. Thus, mathematical proof is essential in higher education for preparing students for academic and professional success.

The importance of mathematical proof in higher education cannot be ignored. This is not only about understanding mathematics theoretically, but also about forming a critical, analytical and logical thinking. Moreover, mathematical proof has significant implications in various fields of science, which makes it an integral part of comprehensive higher education. In this process, mathematical theory develops, and new theorems can be discovered (Velleman, 2019) . In higher education, mathematical proof provides a foundation for the development of students' critical thinking abilities, which are important skills in various scientific disciplines and professions (B. & Bell, 1937) . In addition, mathematical proofs also have wide applications in various scientific disciplines, including computer science, physics, and economics (Strogatz, 2012).

Examples of mathematical proof applications include the Pythagorean theorem, which describes relationships in right-angled triangles, and the prime number theorem, which has broad applications in mathematics and computer science (Macwhinney, 2008). Mathematical proof applications help demonstrate the validity of important concepts. The Pythagorean theorem, for instance, proves the relationship between the sides of a right-angled triangle ($a^2 + b^2 = c^2$), while the prime number theorem provides insights into the distribution of prime numbers, which is essential in areas such as number theory, cryptography, and computer science. Both proofs play crucial roles in advancing mathematical understanding and have real-world applications.

Mathematical proof is not only relevant in pure mathematics, but also plays a role in the development of existing mathematical theories (Hasan, 2020), particularly through real analysis proofs. Mathematicians use proofs to ensure the truth of mathematical statements and to identify the consequences of existing theorems (Krantz, 2017). Apart from the benefits in developing theories, mathematical proofs also play a role in the development of science. Science often utilizes mathematics as a formal language to formulate theories and models that support research (Stroch & Börgers, 2024; Trefethen & Bau, 2022). Without solid mathematical proofs, confidence in the truth of these theories will weaken. The ability to prove mathematics also has a significant value in the world of work. Many jobs in technology, finance, data science, and engineering require the ability to analyze complex problems and formulate solutions systematically (Rota, 2008). Critical thinking skills gained through mathematical proofs can help individuals become more effective problem solvers and contribute to their professional success.

Mathematical proofs can also be a tool of empowerment for students. When students successfully master the art of proof, they feel more confident in facing academic challenges (Abbott, 2015) . This applies not only in mathematics courses, but also in various aspects of their academic life. With a deeper understanding of the central role of mathematical proof in higher education, we can better appreciate its contribution to producing students skilled in analytical thinking, scientific research, and the application of mathematics in a variety of fields.

Mathematical proofs can also be a tool of empowerment for students. When students successfully master the art of proof, they feel more confident in facing academic challenges (Abbott, 2015). This applies not only in mathematics courses, but also in various aspects of their academic life. Unlike previous studies that focus on general proof techniques, this research specifically explores students' abilities in mastering mathematical proofs within the context of real analysis. With a deeper understanding of the central role of mathematical proof in higher education, we can better appreciate its contribution to producing students skilled in analytical thinking, scientific research, and the application of mathematics in a variety of fields.

METHODS

This study employed a qualitative approach with an exploratory research design to gain an in-depth understanding of the process of teaching mathematical proofs in higher education. This approach is chosen to obtain rich insights into the experiences of both students and lecturers in teaching and learning mathematical proofs, as well as to explore the challenges they face during the process. The research participants consist of two primary groups: lecturers and students from mathematics education programs at higher education institutions in Bengkulu Province. Students are selected based on their participation in courses related to mathematical proofs, while lecturers are those who teach courses involving mathematical proofs within the mathematics education program. This selection aims to provide a comprehensive perspective from both groups directly involved in the learning process.

Data collection in this study used three main techniques: in-depth interviews, classroom observations, and document analysis. In-depth interviews were conducted with both lecturers and students to explore their experiences, views, and challenges related to learning and teaching mathematical proofs. The interviews focused on topics such as the teaching approaches used, students' difficulties in understanding proofs, and how mathematical proofs contribute to the development of critical thinking skills. In addition to the interviews, classroom observations were conducted to observe the actual process of teaching mathematical proofs in the classroom. The researcher recorded the interactions between lecturers and students, the teaching techniques applied, and students' responses to the material. This provides a clear picture of the classroom dynamics and how proofs are practically taught.

Furthermore, document analysis was conducted on students' assignments and work related to mathematical proofs. This analysis aims to assess how well students can apply proof concepts in their academic tasks and to evaluate the quality of their logical and critical thinking skills developed through learning mathematical proofs. Data analysis for this research involved thematic analysis. The data collected from interviews, observations, and documents were analyzed to identify key themes related to the teaching and learning of mathematical proofs. The researchers categorized the data based on emerging patterns, such as the challenges students face in learning proofs, the teaching methods employed by lecturers, and the impact of proofs on students' critical thinking skills. Through this approach, the study provides a deeper understanding of the role of mathematical proofs in higher education and offers recommendations for more effective teaching and learning of mathematical proofs.

RESULTS AND DISCUSSION

The findings of this exploratory study provide valuable insights into the teaching and learning of mathematical proofs in higher education, particularly in mathematics education programs in Bengkulu Province. Based on the data collected through in-depth interviews, classroom observations, and document analysis, several key themes emerged related to the process, challenges, and impact of learning mathematical proofs. The findings are organized into the following categories: teaching strategies, challenges faced by students, and the impact of learning mathematical proofs on critical thinking and analytical skills.

Strategies for Teaching Mathematical Proofs

Lecturers employed various teaching strategies to facilitate students' understanding of mathematical proofs. One common approach was active learning, where students were encouraged to engage in group discussions and work collaboratively on proof problems. This method allowed students to learn from each other's insights and made the abstract nature of mathematical proofs more accessible. Mathematical proofs are indeed a fundamental aspect of mathematics, as they are necessary to demonstrate the validity of mathematical statements and theories. The relationship between mathematics and logic is close, with all mathematical constants being logical constants. Mathematics follows from symbolic logic, and all mathematics necessarily follows once the apparatus of logic has been accepted (Russell, 2020). This is a logical process that results in belief in the truth of mathematical statements.

In the world of mathematics, mathematicians use various methods of mathematical proof, and one of the most common methods is the direct proof method, where assumptions are connected to the final result through strong logical steps (Watkins, 2007). For example, to prove that the sum of two even numbers is even, we can start by assuming the two numbers are even and then show that their sum is also even. Mathematical proof is a series of logical arguments that explain the truth of a statement (Syafri, 2017)

Mathematical proofs start with basic axioms and assumptions. Axioms are statements that are accepted without proofs, while basic assumptions are used as premises in proofs (Pólya, 1990) . Mathematical definitions also play an important role in proofs, because they help formulate mathematical statements clearly (Halmos, 1974). In addition, step-by-step explanations of proof techniques were frequently used. Lecturers emphasized breaking down complex proofs into smaller, manageable parts to help students build a logical structure. Socratic questioning was also employed, where lecturers would ask probing questions to guide students toward discovering the proof on their own, promoting critical thinking and independent problem-solving.

Proof by contraposition involves proving a statement by presenting its contraposition (Hardy & Snow, 1992). For example, if we want to prove that if an integer is greater than 7, then it is not an even number. We can prove the contraposition by stating that if it is an even number, then it is less than or equal to 7. Some lecturers also used technology to enhance learning, such as presenting proof concepts through interactive software or video lectures, which allowed students to visualize abstract concepts more concretely. However, despite these efforts, many students expressed difficulties in fully understanding the underlying logic of mathematical proofs.

Important indicators that explicitly express mathematical proofs include recognizing fundamental aspects in mathematics, creating and verifying mathematical conjectures, elaborating and checking arguments and proof in mathematics, and selecting and applying various types of reasoning and proof methods (NCTM, 2000). Three indicators of mathematical proof ability, namely understanding mathematical proof, constructing mathematical proof using various methods such as direct, indirect, or mathematical induction, and evaluating evidence by adding, subtracting, or rearranging a mathematical proof. (Lestari, 2015) . Indicators of mathematical proof ability observed in this research include: 1) ability to read mathematical proof; 2) the ability to carry out mathematical proof using direct, indirect methods, or using mathematical induction; and 3) the ability to criticize proofs by adding, subtracting, or rearranging elements in a mathematical proof.

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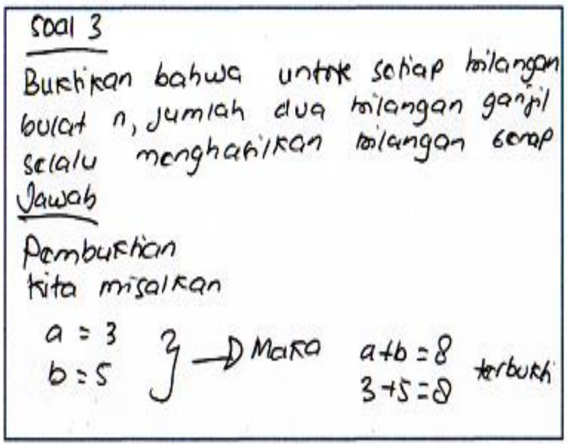
Challenges Faced by Students

The study found that students faced several significant challenges in understanding and applying mathematical proofs. One major challenge identified was abstract thinking. Many students struggled with the transition from performing calculations to constructing logical arguments. This difficulty was particularly evident in more advanced mathematical courses, where students were expected to prove theorems independently.



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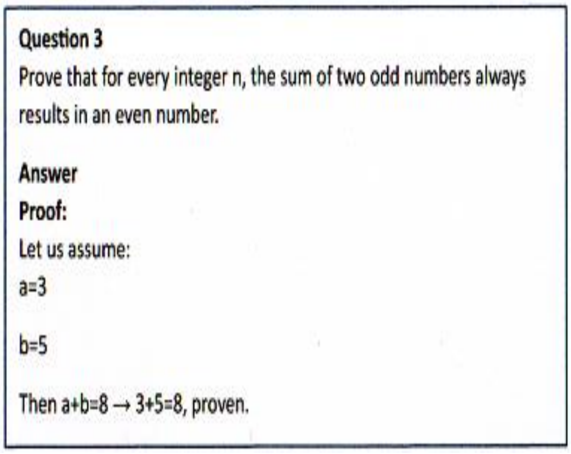

Figure 1. Students' abstract thinking ability

The Figure 1 shown that the students' response reflects difficulties in abstract thinking, as they relied solely on a numerical example (a=3 and b=5) to prove that the sum of two odd numbers is even, without recognizing the need for a generalized mathematical proof. This challenge in abstraction is evident in their inability to use the formal definition of odd numbers, $2k + 1$ where $k \in Z$, to construct a logical argument applicable to all cases. A proper proof requires summing $a = 2k + 1$ and $b = 2m + 1$, resulting in $a + b = 2(k + m + 1)$, which is even because it is divisible by 2. The reliance on specific examples and failure to understand the underlying logical structure highlight their struggle to transition from concrete reasoning to universal generalization.

Abstract thinking in mathematics involves the formalization of empirical concepts and the relationship between abstract-specific mathematical objects and abstract-general concepts. Mathematical objects are self-contained and defined within a system, separate from the real world. Abstraction in mathematics includes ignoring certain features while highlighting others, all while ensuring consistency within the system. The transition to formal thinking in mathematics entails a shift toward axiomatic systems and mathematical proof, moving away from visual representations and symbolic calculations (Mitchelmore & White, 2004; Tall, 2008).

For example, mathematical induction is used to prove statements involving integers and consists of two main steps: the basic step and the induction step (B. & Bell, 1937). This technique is particularly useful for proving statements that exhibit recursive or repeating patterns, such as Gauss's arithmetic theorem. Proof by refutation tests the truth of a statement by attempting to disprove other possible solutions (Alcock & Inglis, 2008). For instance, to prove that the square root of 2 is irrational, we can employ proof by contradiction by assuming the opposite-namely, that the square root of 2 is rational-and subsequently reaching a contradiction.

Mathematical proof also plays an important role in the development of mathematics. When mathematicians construct a proof for a theorem, they not only confirm its truth, but also reveal the relationships and structures within mathematics itself (Strogatz, 2012). This helps deepen understanding of mathematical concepts and leads to the discovery of new theorems. Examples of the use of mathematical proof include the Pythagorean theorem, which expresses relationships in right triangles, and the prime number theorem, which has wide applications in mathematics and computer science (Macwhinney, 2008) . Mathematical proofs are also used in solving complex scientific problems and in developing mathematical models for various practical applications.

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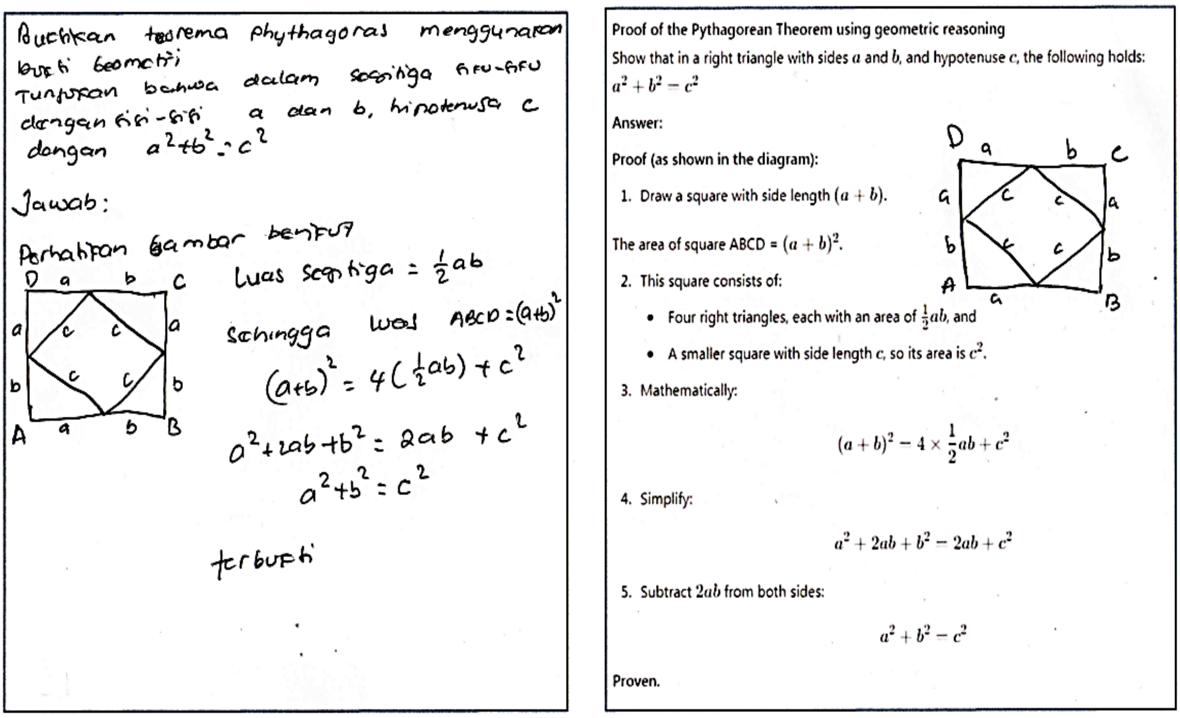


Figure 2. Students' work on the Pythagorean theorem

The students' work demonstrates a solid understanding of the use of mathematical proof in both the Pythagorean theorem (see Figure 2). For the Pythagorean theorem, the students correctly apply a geometric approach to prove the relationship between the sides of a right triangle. Overall, the work effectively shows how mathematical proof is used to explain relationships in geometry and number theory, as well as its applications in real-world problems like cryptography.

It has been demonstrated that using abstract thinking when solving mathematical problems improves students' problem-solving abilities and perceptions of learning. Studies show that when combined with concrete visual representations, abstract visual representations can improve problem-solving performance more than when either type is used alone. Students use less working memory for solving problems when they use abstract visual representations, which also help them concentrate on the important structural aspects of the problem rather than the minutiae. Students can generalize common underlying problem structures and receive perceptual scaffolding and schema activation when abstract and concrete visual representations are combined. Research has indicated that learners who are exposed to both abstract and tangible representations perform better in transfer assessments involving problem-solving (Moreno et al., 2011).

Abstract thinking plays a crucial role in mathematical proofs, as it involves logical reasoning and problem-solving procedures. Students who complete mathematical proofs through intuitive thinking may face difficulties in abstract algebraic proof, which requires analytical thinking (Faizah et al., 2020). Mathematical proofs in university are essential for students to master mathematical argumentative practices. Different proof schemes exist, including informal deductive schemes and formal deductive proof schemes. Students starting their studies at the University of Córdoba in Spain have been found to struggle with spontaneously producing deductive mathematical proofs, highlighting the difficulty in this area. It is important to progressively develop students' knowledge and rationality in applying different proof schemes at different teaching levels (Hanna, 2002; Recio & Godino, 2001; Stefanowicz et al., 2014).

The language of mathematics was also a barrier for some students. They found the formal language and symbols used in mathematical proofs to be confusing, making it harder to express their thoughts and understand the proof structure. Students also noted that time constraints in assignments and exams sometimes led to superficial understanding, where they focused more on completing proofs than on developing deep comprehension. With a deeper understanding of the basic concepts and techniques of mathematical proof, we can appreciate the central role it plays in the development of modern mathematics and its applications in various fields of science (Nonami et al., 2019).

Impact of Learning Mathematical Proofs on Critical Thinking and Analytical Skills

Despite the challenges, the study revealed that learning mathematical proofs has a positive impact on students' critical thinking and analytical skills. Most students acknowledged that the process of constructing and verifying proofs helped them develop a more logical and structured approach to problem-solving. One of the key benefits was the improvement in analytical reasoning. Students reported that they became better at breaking down problems into smaller parts, identifying patterns, and drawing logical conclusions.

Mathematical proofs play an important role in higher education, especially in developing a deep understanding of mathematics in students. In higher education, mathematical proof is not only emphasized as a technical skill, but also as a tool to understand mathematical concepts in more depth and hone critical thinking skills (Alcock & Inglis, 2008).

One important aspect in learning mathematical proofs in higher education is the application of basic proof concepts, such as axioms, definitions and theorems, in various branches of mathematics. (Halmos, 1974) . Students are encouraged to understand the basics of logic used in mathematical proof, such as implication, contraposition, and proof by contradiction. This helps them build a strong foundation for analytical thinking (Velleman, 2019) .

The process of proving mathematical statements also contributed to the development of self-confidence. As students successfully completed proofs, they reported feeling more confident in their ability to solve complex mathematical and real-world problems. This confidence extended beyond mathematics and was applicable to other areas of academic work, such as writing research papers and soving complex problems in various disciplines.

Mathematical proofs also provide an understanding of various proof techniques, including direct proof, proof by contraposition, and proof by mathematical induction (Tao, 2006; Watkins, 2007). Mastering these various proof techniques helps students solve complex mathematical problems and develop critical thinking skills (Hardy & Snow, 1992).

Mathematical proofs also have an important role in developing accuracy and neatness in thinking and communicating (Strogatz, 2012). Students are encouraged to express their thoughts systematically and clearly in the form of precise mathematical writing. This ability is valuable in a variety of contexts, including scientific research and mathematical model development (Čížková & Čížek, 2012). In addition, mathematical proofs prepare students for advanced courses and research in mathematics (B. & Bell, 1937). In advanced courses, students will be involved in proving more complex and in-depth theorems. A strong understanding of mathematical proof provides a strong foundation for their academic progress (Nonami et al., 2019).

In the context of research, mathematical proofs also have a key role (Alexanderson & Polya, 1979). Mathematicians and computer scientists often use mathematical proofs to support or validate their findings. This helps ensure correctness and accuracy in the development of mathematical models and complex solutions (Macwhinney, 2008). Mathematical proof is not only about formulating formal proofs, but also about developing strong mathematical intuition (Apostol, 1991) . Students in higher education are invited to dig deeper into mathematical concepts and connect them to real world problems (Krantz, 2017). Mathematical proofs in higher education are not only an integral part of the mathematics curriculum, but also a powerful tool for developing analytical thinking, critical thinking skills, and a deeper understanding of the world of mathematics and science (Ross, 2013).

Observations from Classroom Practices

Classroom observations confirmed the findings from the interviews, showing that active engagement and collaboration among students played a critical role in their understanding of proofs. During group activities, students were more willing to ask questions and express their reasoning, often guiding each other to the correct conclusions. Lecturers also provided real-time feedback during these activities, which helped clarify misunderstandings and reinforce correct reasoning.

Through proof assignments, lecturers can see: (1) how students are able to argue logically, (2) how students use examples and counterexamples to support their arguments, (3) what weaknesses students experience in reasoning, and (4)) what misconceptions students often experience. (Jamilah & Fadillah, 2017) . Lecturers can assess students' ability to argue logically, use examples and counterexamples to support arguments, identify weaknesses in reasoning, and uncover misconceptions that students often experience. Moreover, the ability to construct rigorous arguments in mathematics was seen as beneficial for students pursuing careers in fields such as engineering, computer science, economics, and data science, where logical reasoning and problem-solving are highly valued.

Thus, mathematical proofs in college are not only an important part of the mathematics curriculum, but also serve as a highly effective tool for developing analytical thinking abilities, critical thinking skills, and a deeper understanding of the world of mathematics and science.

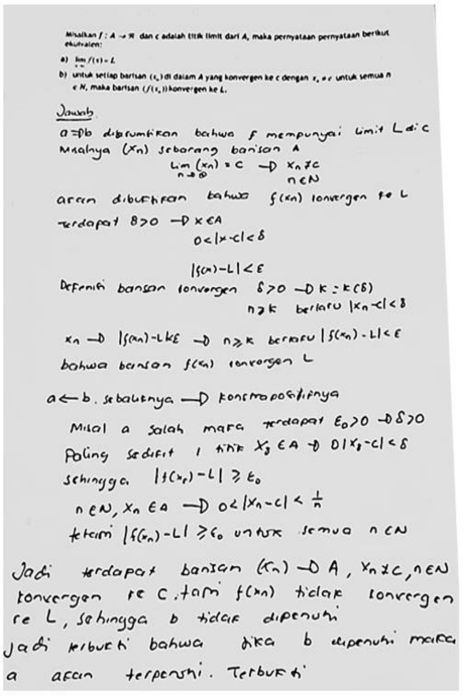
Document Analysis: Students' Work and Assignments

The analysis of students' assignments revealed a mixed level of proficiency in mathematical proofs. Many students struggled with writing clear, logical, and complete proofs, often missing crucial steps or failing to justify their reasoning adequately. However, students who participated more actively in classroom discussions and engaged with supplementary learning resources demonstrated a higher level of competence in their proofs.

Assignments also indicated that students often resorted to memorizing proof templates rather than developing a deep understanding of the underlying logic. This suggests the need for teaching strategies that emphasize not just the mechanics of proofs but also the conceptual understanding and logical connections that underpin them.

This question tests students' understanding of the concepts of function limits and sequence convergence, which are important topics in mathematical analysis. Students are expected to link two equivalent statements: that if $f : A \to \Re$ and c is a limit point of A, then the function f has a limit L at c if and only if for every sequence (xn) in A that converges to c, the sequence $f(x\_n)$ converges to L (Bartle & Sherbert, 2000). In solving this problem, students will use critical thinking skills by constructing logical arguments and providing in-depth proofs of the relationship between these statements. They are also required to carefully analyze fundamental concepts and apply their understanding to broader situations, thereby strengthening their analytical abilities in solving more complex mathematical problems.

The research findings indicate that critical thinking and analytical skills play a significant role in mathematical proofs, with clear differences observed across students with high, medium, and low abilities. High-ability students demonstrate well-structured, logical, and precise solutions to mathematical proofs. They effectively apply formal definitions, construct mathematical arguments, and validate their conclusions with thorough reasoning, showcasing advanced critical thinking skills (see Figure 3).



Misalkan $f: A \to \mathbb{R}$ dan c adalah titik limit dari A, maka pernyataan pernyataan berikut ekuivalen:

a) $\lim f(x) = L$

b) untuk setiap barisan (x_n) di dalam A yang konvergen ke c dengan $x_n \neq c$ untuk semua $n \in N$, maka barisan $(f(x_n))$ konvergen ke L.

Jawab

a ⇒ b dibuktikan bahwa f mempunyai limit L di c
Misalnya (x_n) sebarang barisan A
$\lim_{n\to\infty} (x_n) = c \to x_n \neq c$, $n \in N$
akan dibuktikan bahwa $f(x_n)$ konvergen ke L
terdapat $\delta > 0 \to x \in A$
$0 < |x - c| < \delta$
$|f(x) - L| < \varepsilon$
Definisi barisan konvergen $\delta > 0 \to K = K(\delta)$
$n \geq K$ berlaku $|x_n - c| < \delta$
$x_n \to |f(x_n) - L| < \varepsilon \to n \geq K$ berlaku $|f(x_n) - L| < \varepsilon$
bahwa barisan $f(x_n)$ konvergen L
a ⇐ b. sebaliknya → kontrapositifnya
Misal a salah maka terdapat $\varepsilon_0 > 0 \to \delta > 0$
paling sedikit 1 titik $x_\delta \in A \to 0 < |x_\delta - c| < \delta$
sehingga $|f(x_\delta) - L| \geq \varepsilon_0$
$n \in N, x_n \in A \to 0 < |x_n - c| < \frac{1}{n}$
tetapi $|f(x_n) - L| \geq \varepsilon_0$ untuk semua $n \in N$
Jadi terdapat barisan $(x_n) \to A$, $x_n \neq c, n \in N$ konvergen ke c, tapi $f(x_n)$ tidak konvergen ke L, sehingga b tidak dipenuhi
Jadi terbukti bahwa jika b dipenuhi maka a akan terpenuhi. Terbukti

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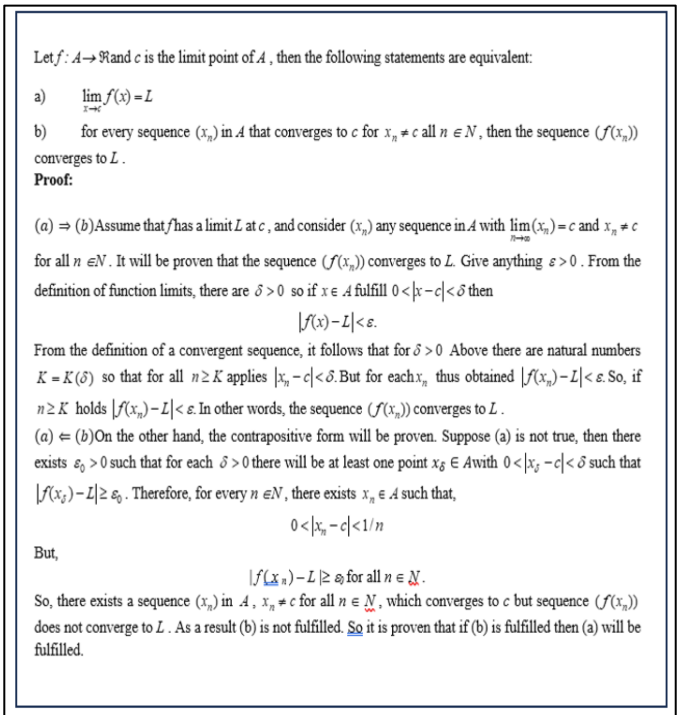


Let $f: A \to \Re$ and c is the limit point of A, then the following statements are equivalent:

a) $\lim_{x\to c} f(x) = L$

b) for every sequence (x_n) in A that converges to c for $x_n \neq c$ all $n \in N$, then the sequence $(f(x_n))$ converges to L.

Proof:

$(a) \Rightarrow (b)$ Assume that f has a limit L at c, and consider (x_n) any sequence in A with $\lim_{n\to\infty}(x_n) = c$ and $x_n \neq c$ for all $n \in N$. It will be proven that the sequence $(f(x_n))$ converges to L. Give anything $\varepsilon > 0$. From the definition of function limits, there are $\delta > 0$ so if $x \in A$ fulfill $0 < |x - c| < \delta$ then

$$|f(x) - L| < \varepsilon.$$

From the definition of a convergent sequence, it follows that for $\delta > 0$ Above there are natural numbers $K = K(\delta)$ so that for all $n \geq K$ applies $|x_n - c| < \delta$. But for each x_n thus obtained $|f(x_n) - L| < \varepsilon$. So, if $n \geq K$ holds $|f(x_n) - L| < \varepsilon$. In other words, the sequence $(f(x_n))$ converges to L.
$(a) \Leftarrow (b)$ On the other hand, the contrapositive form will be proven. Suppose (a) is not true, then there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$ there will be at least one point $x_\delta \in A$ with $0 < |x_\delta - c| < \delta$ such that $|f(x_\delta) - L| \geq \varepsilon_0$. Therefore, for every $n \in N$, there exists $x_n \in A$ such that,

$$0 < |x_n - c| < 1/n$$

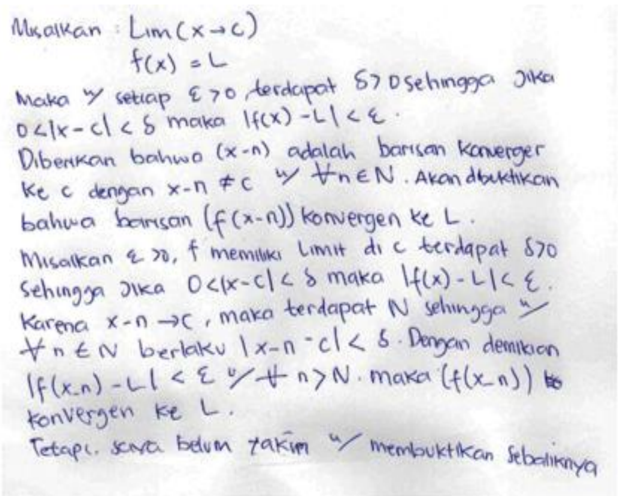
But,

$$|f(x_n) - L| \geq \varepsilon_0 \text{ for all } n \in N.$$

So, there exists a sequence (x_n) in A, $x_n \neq c$ for all $n \in N$, which converges to c but sequence $(f(x_n))$ does not converge to L. As a result (b) is not fulfilled. So it is proven that if (b) is fulfilled then (a) will be fulfilled.

Figure 3. Answers from high-ability students'

In contrast, medium-ability students often exhibit partial understanding, initiating proofs correctly but struggling to complete them or provide adequate justifications (Figure 4). This reflects a limited depth in their critical thinking and analytical abilities.

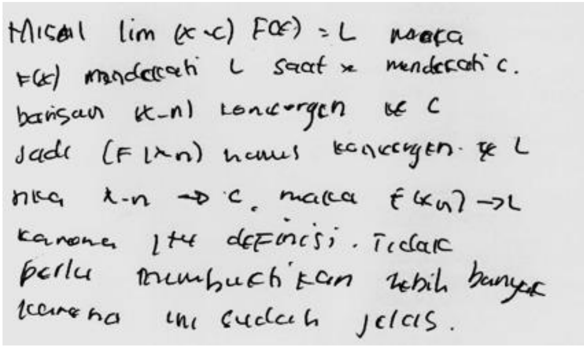


Misalkan : $\lim(x \to c)$ $f(x) = L$
Maka ∀ setiap $\varepsilon > 0$, terdapat $\delta > 0$ sehingga jika $0 < |x - c| < \delta$ maka $|f(x) - L| < \varepsilon$.
Diberikan bahwa (x-n) adalah barisan konverger ke c dengan $x\text{-}n \neq c$ ∀ $\forall n \in N$. Akan dibuktikan bahwa barisan (f(x-n)) konvergen ke L.
Misalkan $\varepsilon > 0$, f memiliki limit di c terdapat $\delta > 0$ sehingga jika $0 < |x - c| < \delta$ maka $|f(x) - L| < \varepsilon$.
Karena $x\text{-}n \to c$, maka terdapat N sehingga ∀ $\forall n \in N$ berlaku $|x\text{-}n - c| < \delta$. Dengan demikian $|f(x\_n) - L| < \varepsilon$ ∀ $\forall n > N$. maka (f(x_n)) konvergen ke L.
Tetapi, saya belum yakin ∀ membuktikan sebaliknya

English version:

Suppose $\lim(x \to c) f(x) = L$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Given that (x_n) is a sequence that converges to c with $x\_n \neq c$ for all $n \in N$, it will be proven that the sequence $(f(x\_n))$ converges to L. Suppose $\varepsilon > 0$. Since f has a limit at c, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Since $x\_n \to c$, there exists N such that for all $n > N, |x\_n - c| < \delta$. Therefore, $|f(x\_n) - L| < \varepsilon$ for all n>N. Thus, $(f(x\_n))$ converges to L. However, I am unsure how to prove the converse part.

Figure 4. Answer from moderate-ability student'


Misal lim (x→c) F(x) = L maka
F(x) mendekati L saat x mendekati c.
barisan (x_n) konvergen ke c
Jadi (F(x_n)) harus konvergen ke L
jika x_n → c, maka f(x_n) → L
karena itu definisi. Tidak
perlu membuktikan lebih banyak
karena ini sudah jelas.

English version:

Suppose $\lim(x \to c) f(x) = L$. Then, $f(x)$ apporaches L as x approaches c. The sequence (x_n) converge to L. If $x\_n \to c$, then $f(x\_n) \to L$, because that is the definition of a limit. There is no need to prove further, because it is already clear.

Figure 5. Answers from low-ability students'

Low-ability students (Figure 5), on the other hand, face significant challenges in grasping basic concepts and applying them to proofs. Their answers tend to lack structure and fail to address most of the required steps, highlighting the impact of insufficient critical thinking skills on their overall performance. In conclusion, the study emphasizes the strong correlation between critical thinking skills and success in mathematical proofs. Students with higher critical thinking abilities perform better due to their ability to analyze and reason effectively. Thus, incorporating teaching strategies that foster critical thinking, such as inquiry-based learning, is essential to improving mathematical understanding and academic achievement across all levels of ability.

Based on the research findings critical thinking skills and analytical abilities play a crucial role in mathematical proofs. Students' critical and mathematical thinking skills are related, and both increase as academic achievement and critical thinking skills increase. Gender does not significantly impact critical and mathematical thinking skills, but grade level does. Students' critical skills and academic achievement explain 25.0% of the total variance of mathematical thinking skills (Er, 2024).

Critical thinking skills and analytical abilities play a crucial role in mathematical proofs. According to Fisher & Scriven (2001) , critical thinking skills involve interpreting, analyzing, and evaluating ideas, arguments, and observations, as well as thinking about assumptions and drawing out implications. Additionally, reasoning, argumentation, and proof are essential concepts in mathematical practice, with logic being closely associated with mathematics (Berki & Valtanen, 2007; Hanna, 2020).

Visualization techniques and structured approaches, such as organizing knowledge around key components like definitions and theorems, can enhance analytical abilities in mathematical proofs. The lack of time and changes in course literature can impact the focus on proof techniques in mathematical practice (Hemmi, 2006; Muzangwa & Ogbonnaya, 2024).

CONCLUSION

Mathematical proof plays a crucial role in higher education, particularly in developing critical thinking skills, advancing scientific research, and applying mathematics across various disciplines. In this context, each mathematical proof discussed in this study pertains to real analysis. Through the process of proving mathematical statements in real analysis, students not only determine truth but also gain a deeper understanding of mathematical structures and explore new concepts. This process trains students to think logically, analytically, and critically. These skills are essential for solving academic challenges and excelling in professional careers.

Moreover, mathematical proofs in real analysis are important for advancing new theories and scientific research. They provide a solid foundation for a deeper understanding of mathematics, which in turn enriches various fields, including physics, economics, computer science, and engineering. The ability to prove mathematical statements ensures the validity of existing theorems and contributes to the progress of science as a whole. Beyond its role in pure mathematics, the skill of proving statements is also invaluable in the workforce. The analytical skills developed through mathematical proofs in real analysis make students more effective problem solvers, a trait highly sought after in industries such as technology, data science, and finance. By gaining a deeper understanding of the central role of mathematical proof in higher education, particularly in real analysis, we can better appreciate their contribution to producing students skilled in analytical thinking, scientific research, and the practical application of mathematics across various fields. Mathematical proof in real analysis is not only a foundation for learning mathematics but also a gateway to broader academic and professional success.

The findings of this study highlight significant differences in the critical thinking and analytical abilities of students with high, medium, and low levels of ability. High-ability students demonstrate a clear mastery of mathematical proofs, producing logical, well-structured, and precise solutions. Their advanced critical thinking skills enable them to apply formal definitions, analyze problems thoroughly, and construct sound mathematical arguments to validate their conclusions. Medium-ability students, while showing partial understanding, often struggle to complete proofs or provide adequate justifications. They are able to initiate proofs correctly but lack the depth of critical thinking needed to address complex steps or refine their reasoning. Low-ability students face major challenges in understanding and applying basic concepts, leading to unstructured and incomplete proofs. This reflects a lack of critical and analytical thinking skills, which hinders their ability to engage in the logical reasoning required for mathematical problem-solving.

This study underscores the strong correlation between critical thinking skills and success in constructing mathematical proofs. Students with higher levels of critical and analytical thinking excel in solving mathematical problems and navigating complex reasoning processes. To address the disparities among students, particularly those in the medium and low-ability categories, it is essential to integrate teaching strategies that foster critical thinking and analytical skills, such as inquiry-based learning and guided problem-solving. These approaches can help bridge the gaps in students' abilities, ensuring a more equitable development of mathematical competence. Future research should further explore how tailored instructional strategies can improve students' critical thinking and analytical reasoning, enhancing their academic outcomes and readiness for professional challenges.

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DECLARATIONS

Author Contribution	:	DI: Conceptualization, Writing - Original Draft, Editing and Visualization, Formal analysis and Writing — review & editing; AS: Validation, Supervision, Writing - Review & Editing, and Methodology; DR: Validation, Supervision, Writing - Review & Editing, and Methodology; FSS: Writing — original draft, Project administration and Formal analysis,
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