Mathematics Education Journal
ISSN: 2549-1040 (e) | 1978-0044 (p)
https://doi.org/10.22342/mej.v19i2.pp389-412

Exploring the Mathematical Reasoning Skills Across Different Levels of Prospective Mathematics Teachers: A Mixed-Methods Investigation

Dewi Hamidah[1,2], Susiswo[1,*], Hery Susanto[1], Zun Azizul Hakim[3], Sharifah Osman[4]

[1]Mathematics Education Department, Universitas Negeri Malang, Malang, Indonesia
[2]Mathematics Education Study Program, Institut Agama Islam Negeri Kediri, Kediri, Indonesia
[3]Islamic Psychology Study Program, Universitas Islam Negeri Sayyid Ali Rahmatullah, Tulungagung, Indonesia
[4]Department of Science and Mathematics, Universiti Teknologi Malaysia, Johor Bahru, Malaysia
*Email: susiswo.fmipa@um.ac.id

Abstract

Mathematical reasoning is a fundamental competence for prospective mathematics teachers, as it underpins their ability to interpret, extend, and respond to students' mathematical thinking. However, there is a documented gap in how these reasoning skills evolve across the stages of teacher education, particularly in relation to the instructional experiences and learning environments provided throughout the teacher education program. Existing research has not sufficiently addressed how reasoning abilities develop over time within the same academic trajectory, creating a need to explore longitudinal variations and instructional influences. This study investigates the differences of mathematical reasoning skills—specifically conjecturing, generalizing, and justifying—among prospective mathematics teachers across different semesters of their undergraduate education. Employing an exploratory mixed-methods approach, the research involved 198 undergraduate students enrolled in a Mathematics Education program in Kediri, Indonesia. Data were collected through two validated numeracy tasks and analyzed using qualitative methods to identify reasoning patterns, complemented by quantitative analysis via the Kruskal-Wallis test to examine semester-based differences. The findings reveal that while students generally exhibit strength in conjecturing, persistent challenges remain in generalizing and justifying mathematical ideas, with more advanced students demonstrating comparatively higher proficiency. These results underscore the necessity for early, sustained, and scaffolded interventions within mathematics teacher education programs to nurture comprehensive reasoning capabilities. The study offers critical insights into the developmental trajectory of mathematical reasoning and informs the design of curriculum frameworks that better support the intellectual growth of future mathematics educators.

Keywords: Mathematical Reasoning, Numeracy Problems, Prospective Mathematics Teacher, Semester-based Differences

How to Cite: Hamidah, D., Susiswo, Susanto, H., Hakim, Z. A., & Osman, S. (2025). Exploring the Mathematical Reasoning Skills Across Different Levels of Prospective Mathematics Teachers: A Mixed-Methods Investigation. *Mathematics Education Journal*, *19*(2), 389–412. https://doi.org/10.22342/mej.v19i2.pp389-412

INTRODUCTION

Mathematical reasoning constitutes a fundamental cognitive competency that underlies essential intellectual processes such as decision-making, problem-solving, and critical thinking. It holds a pivotal position within mathematics education and is widely acknowledged as a vital element of 21st-century competencies necessary for addressing multifaceted and authentic real-world problems (Bragg et al., 2016; Oliver, 2021). Empirical evidence from correlational studies further substantiates the role of mathematical reasoning as a strong predictor of students' mathematical achievement (Singley & Bunge, 2014). By facilitating comprehension, the evaluation of information, and the formulation of logical judgments, mathematical reasoning contributes significantly to students' sustained engagement and achievement in mathematics learning (Bragg et al., 2016). Consequently, fostering mathematical reasoning skills from an early stage is imperative to preparing learners for both academic excellence and success in their future professional and everyday lives.

Received December 5, 2024; Revised April 24, 2025; Accepted April 30, 2025

The development of reasoning skills entails the ability to trace and critically assess chains of argumentation, comprehend the epistemological basis of mathematical evidence as distinct from other forms of reasoning, identify core mathematical principles within arguments, and construct both formal and informal justifications (Gürbüz & Erdem, 2016; Niss, 2003). Within the domain of mathematics, reasoning fulfills multiple functions, including but not limited to verification, explanation, systematization, exploration, communication, and theory construction (Hwang et al., 2017; Yackel & Hanna, 2003). As an indispensable component of mathematical proficiency, it enhances learners' conceptual understanding and supports deeper engagement with mathematical ideas (Bragg et al., 2016). Accordingly, it is imperative that both learners and educators in mathematics acquire and continuously refine their mathematical reasoning abilities.

Mathematical reasoning is a foundational element of mathematics and, as such, plays a critical role in the learning and teaching of mathematics in school contexts (Brodie et al., 2010). This emphasis aligns with the Programme for International Student Assessment (PISA), which highlights mathematical process competencies—such as problem-solving, modelling, and reasoning—as central to mathematics education (Hjelte et al., 2020). Given that the depth and quality of an individual's reasoning are shaped by their conceptual understanding, mathematical reasoning becomes a key mechanism through which learners make sound decisions (Gürbüz & Erdem, 2016). Therefore, the cultivation of mathematical reasoning skills is essential for nurturing learners' critical thinking capacities and enhancing their ability to make informed and logical decisions. In light of its growing recognition as a core mathematical competence, it is imperative to investigate the role of reasoning in supporting both instructional practices and student learning in mathematics education.

Within this educational context, mathematical reasoning serves not only as a tool for problem-solving but also as a vehicle for expressing and translating mathematical ideas into pedagogically meaningful representations (Hwang et al., 2017; Yackel & Hanna, 2003). PISA's evolving framework reflects a broader assessment approach, incorporating a range of competencies such as reasoning, argumentation, and the use of mathematical symbols and formalism, all of which are essential for solving complex, real-world problems (Niss, 2015; Pettersen & Braeken, 2019). Furthermore, contemporary assessment practices increasingly prioritize learners' ability to apply mathematical knowledge in authentic contexts over mere recall of procedures or memorization of facts (Girardin et al., 2019). As a result, the development of mathematical reasoning has become a focal point in both instructional design (Herbert & Bragg, 2021) and student learning trajectories (Wong & Low, 2020). The abilities to generalize mathematical structures and to justify solutions underpin logical reasoning and effective problem-solving (Widjaja et al., 2021). For prospective mathematics educators, mastering reasoning skills is particularly crucial, as these competencies empower them to facilitate deep mathematical understanding and guide their students toward meaningful engagement with mathematics.

Despite the well-documented significance of mathematical reasoning, considerable challenges persist in effectively nurturing these skills, particularly among prospective mathematics teachers. These individuals are expected not only to grasp sophisticated mathematical ideas but also to facilitate learning environments that promote reasoning and critical thinking among their future students (Brodie et al., 2010). However, the development of mathematical reasoning skills is not uniform across educational levels or academic trajectories. Ding (2018) noted that while mathematical reasoning generally shows a positive progression across grade levels, the pace and nature of development vary depending on the specific sub-skills involved. At the tertiary level, such variation becomes more pronounced, as reasoning skills tend to differ based on students' academic majors and institutional contexts (Ding et al., 2016). These discrepancies underscore the need for a systematic investigation into how mathematical reasoning evolves throughout teacher education programs.

Findings from longitudinal research further illuminate the complexities associated with the progression of reasoning skills. For example, Küchemann and Hoyles (2006), in their analysis of high-achieving students' mathematical reasoning in algebra and geometry, identified only moderate improvements over time, with students' responses exhibiting variability influenced by curriculum reforms. Similarly, El Mouhayar (2018) reported a gradual advancement in students' levels of reasoning and generalization across both numerical and figural reasoning tasks, suggesting differentiated developmental pathways. These findings raise critical concerns regarding the adequacy of teacher preparation programs in equipping prospective educators with the necessary reasoning competencies to foster deep mathematical understanding and thinking in school settings.

In addition to developmental disparities, there are significant pedagogical challenges and instructional limitations that impede the advancement of mathematical reasoning within teacher education programs. For instance, prospective early childhood educators working with children aged 0–3 often prioritize the identification of mathematical concepts over recognizing and interpreting children's reasoning processes (Vanegas et al., 2021). In the context of primary education, while preservice teachers are generally proficient in eliciting student thinking, they often require targeted support to effectively scaffold and deepen students' mathematical reasoning (Shure & Liljedahl, 2024). At the middle school level, prospective teachers may demonstrate the ability to anticipate students' reasoning strategies but frequently lack the skills to identify and respond to pivotal reasoning moments within instructional interactions (Simsek, 2025). Moreover, a persistent reliance on rule-based instruction and dichotomous understandings of correctness constrains learners' opportunities to engage conceptually and reason independently (Ryken, 2009). To mitigate these issues, pedagogical frameworks that emphasize mathematical processes such as generalization, justification, and classification have been shown to support more effective reasoning-based instruction (Rodrigues et al., 2021). Furthermore, strategic interventions that enhance prospective teachers' planning, monitoring, and reflective practices are essential to cultivating reasoning-focused teaching approaches (Mendes et al., 2022). Despite a growing body of research on school students and practicing teachers, empirical studies examining the development of mathematical reasoning among preservice teachers—particularly across the various stages of undergraduate education—remain relatively scarce. This is a critical gap, considering the strong correlation between preservice teachers' reasoning competencies and their future instructional efficacy in mathematics classrooms.

The existing literature underscores this research gap. Although substantial attention has been devoted to mathematical reasoning among K–12 learners and in-service educators, relatively limited investigation has explored the progression of reasoning abilities in preservice teachers across different phases of their academic preparation (Bragg et al., 2016; Herbert et al., 2015). This oversight is particularly concerning, given the strong interdependence between the reasoning skills of future teachers and the quality of mathematics instruction they will eventually provide. Additionally, empirical studies have revealed that both students and teachers frequently exhibit underdeveloped mathematical argumentation skills, such as generalizing and justifying, which can hinder deeper learning and conceptual understanding (Melhuish et al., 2020). In response to this gap, the present study aims to examine how mathematical reasoning skills—particularly in the form of generalization and justification—evolve across semesters within mathematics teacher education programs. The study specifically investigates how these competencies emerge and are expressed in the context of problem-solving, with a focus on numeracy-based tasks.

The cultivation of mathematical reasoning is vital to the effective teaching and learning of mathematics, as it enhances students' conceptual understanding and their capacity to apply mathematical knowledge in real-world contexts (Mukuka et al., 2023; Simsek, 2025). Teachers who are proficient in fostering reasoning play a pivotal role in enabling learners to develop these higher-order thinking skills (Mukuka et al., 2023). Nevertheless, empirical evidence highlights persistent challenges encountered by prospective mathematics teachers, particularly during the critical phases of task planning, exploration, and monitoring that are intended to promote mathematical reasoning (Mendes et al., 2022). These challenges often stem from limited proficiency in mathematical language and difficulty in recognizing instances of reasoning in children's mathematical thinking (Bragg et al., 2016; Vanegas et al., 2021). Additionally, prospective teachers frequently exhibit difficulties in solving numeracy problems, employing diverse problem-solving strategies, and engaging in both algorithmic and creative reasoning processes (Palengka et al., 2022). The development of their reasoning capabilities is also influenced by the quality and type of educational experiences and instructional materials encountered during their training (Nhiry et al., 2023). Despite these findings, limited research has systematically examined how mathematical reasoning skills evolve across semesters in teacher education programs. This gap signals the need for a more comprehensive investigation into the reasoning competencies of prospective mathematics teachers at various academic stages, especially in relation to numeracy-based problem contexts.

This study aims to examine the mathematical reasoning abilities of prospective mathematics teachers at different levels of their undergraduate education. Specifically, it investigates how these individuals formulate and validate conjectures, generalize mathematical relationships, and construct logical justifications in response to numeracy tasks. The research further seeks to determine whether significant differences exist in reasoning performance based on the participants' semester level. To this end, the study will assess three critical aspects of reasoning: conjecturing, generalizing, and justifying, utilizing numeracy problems as the central instrument for evaluation.

The primary contribution of this study lies in its comparative analysis of mathematical reasoning across semester levels—an area that has received limited attention in prior research. By examining how reasoning skills are acquired and expressed over the course of teacher preparation, the study aims to generate actionable insights that can inform the design and refinement of teacher education curricula. The research is delimited to undergraduate students enrolled in mathematics education programs, with an exclusive focus on individuals preparing to become mathematics teachers. The findings are expected to contribute meaningfully to the broader discourse in mathematics education by identifying specific areas in which prospective teachers may require targeted instructional support to enhance their mathematical reasoning capabilities.

METHODS

In this study, an exploratory mixed-methods research design was purposefully adopted to examine the comprehension of numeracy problem-solving among prospective mathematics teachers. The investigation commenced with a qualitative phase, which served as the foundational basis for subsequent quantitative analysis, as recommended by Creswell (2014). During the initial stage, a content analysis was performed on participants' written responses to a set of numerical problems to identify key features of their mathematical reasoning.

The qualitative findings were subsequently transformed into quantifiable data to enable a more nuanced examination of differences in reasoning abilities across various semester cohorts, consistent with the explanatory sequential approach outlined by Creswell and Clark (2011). The analysis concentrated on three core aspects of mathematical reasoning: conjecturing, generalizing, and justifying. These aspects collectively provided a comprehensive measure of participants' overall mathematical reasoning proficiency. The procedural framework guiding this study is illustrated in Figure 1.

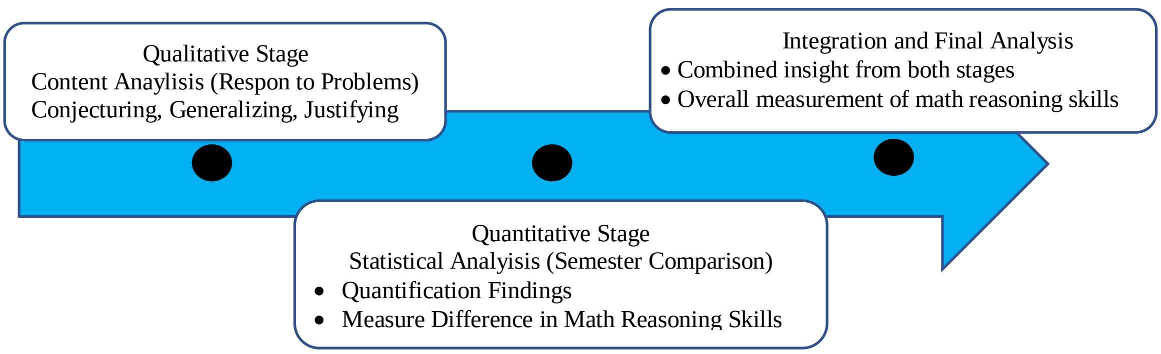


Figure 1. Sequential mix-method research framework

The research instrument comprised two numeracy-based problems, accessible through the provided link. To ensure its validity, the instrument underwent expert evaluation by four mathematics education lecturers, each holding a master's degree and possessing an average of eight years of teaching experience at the university level. These experts reviewed the instrument's content and language, ensuring its alignment with the essential components of mathematical reasoning—namely, conjecturing, generalizing, and justifying. The validation process confirmed that the instrument appropriately measured the intended constructs, establishing its reliability for assessing mathematical reasoning in this study.

The participants consisted of 198 undergraduate students enrolled in the Mathematics Education program at an Islamic State University in East Java, Indonesia. The sample was stratified across three academic levels—first, third, and fifth semesters—designated as Group 1, Group 2, and Group 3, respectively, with 66 students randomly selected for each group. The selection ensured that students had experienced similar instructional conditions within their respective semesters. The study primarily aimed to investigate variations in mathematical reasoning abilities across these groups in the context of numeracy problem-solving. Participants were further classified into three performance categories—high, medium, and low—based on their scores on a mathematical reasoning assessment. Representative individuals from each category were selected for follow-up interviews to gain deeper insights into their reasoning processes.

During scheduled lecture hours, prospective mathematics teachers were instructed to solve two numeracy problems embedded in personal and contextual scenarios. Clear, detailed instructions were provided for each item, outlining the expectations for complete and well-structured written responses. The allotted duration for task completion was 100 minutes, equivalent to two lecture hours, a time frame considered sufficient for maintaining student concentration and engagement throughout the assessment.

Upon task completion, a subset of participants was selected for follow-up interviews to gain deeper insights into their mathematical reasoning processes, thereby enriching the qualitative dimension of the study. The written responses were subsequently evaluated by the research team using a predefined scoring rubric. Each script was analyzed both qualitatively and quantitatively, based on the established criteria for assessing mathematical reasoning. The scores were categorized according to the rubric presented in Table 1.

To determine whether statistically significant differences existed in mathematical reasoning abilities across the three groups (first, third, and fifth semesters), a Kruskal–Wallis H test was employed. This non-parametric test was chosen due to the ordinal nature of the data and the assumption of non-normal distribution. The following section presents the participants' performance in solving the two numeracy problems, which are accessible via the following link: [https://s.id/aNrup]. These problems were developed with reference to numeracy indicators from the Programme for International Student Assessment (PISA) and Indonesia's National Assessment of Minimum Competency (*AKM Kemdikbud*). Table 1 outlines the assessment indicators and scoring allocations for each component of mathematical reasoning.

Table 1. Description of the indicators and scores of each aspect of mathematical reasoning

Indicators	Max. Score
Aspect: Conjecturing	
No answer or all the conjecture is incorrect.	0
Capable of utilizing available data but unable to select rules to predict the answer and the desired solution process.	5
Make predictions and solve problems involving number patterns and object configurations.	10
Capable of creating custom rules to arrive at accurate answer predictions and processing solutions based on data provided.	15
Aspect: Generalizing	
No answer.	0
Can comprehend established patterns and relationships but unable to select a set of rules to analyze mathematical situations, draw analogies, and generalize.	5
Analyze mathematical situations, draw analogies, and generalize to number pattern material using patterns and relationships.	15
Can develop own set of rules to analyze mathematical situations, draw analogies, and generalize based on data provided.	20
Aspect: Justifying	
No Justification.	0
Only capable of utilizing the information provided, but not yet capable of selecting a set of rules to prepare valid arguments.	5
Can develop valid arguments on the material of number patterns and object configuration by explaining the strategy selected and implemented as well as the reasons it worked or not.	15
Capable of utilizing the information provided to develop their own set of rules to construct valid arguments.	20

RESULTS AND DISCUSSION

Mathematical Reasoning Skills of Prospective Teacher

Prospective Teacher with High Category

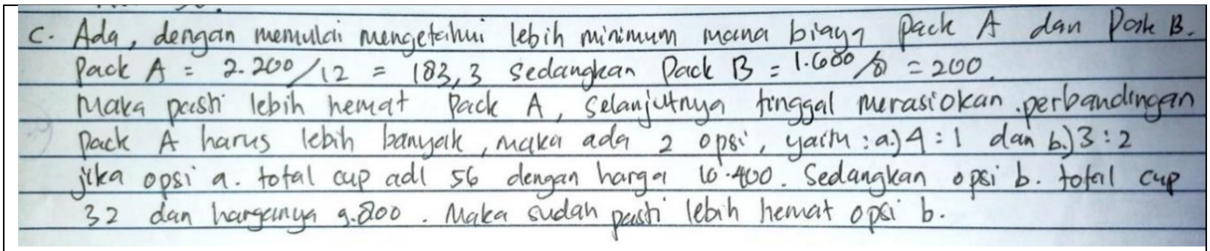
Initially, we examined the prospective teachers' mathematical reasoning, which encompassed three key aspects: conjecturing, generalizing, and justifying. The responses from the three groups of prospective teachers, classified under the high mathematical reasoning category, are presented as follows. In the domain of conjecturing, all three groups exhibited competence. They developed rules to predict specific outcomes and engaged in solution processes based on the provided data. The methodology employed by the prospective teachers across all groups involved trial and error, or a direct attempt to identify the correct solution. The conjectures were articulated either in the form of tables or equations. Transcript 1 provides an excerpt from an interview related to the conjecturing aspect, specifically focusing on the responses from participants in Group 1 (H1).

Transcript 1:

Researcher: Explain how you estimate how many packs of A or B should be purchased?
H1 : The problem states that the cost of pack A is IDR 2,200, translating to a cost of IDR 183.33 per cup. Meanwhile, pack B contains 8 cups and is priced at IDR 1,600, therefore resulting in a cost of IDR 200 per cup. However, since the cup is not sold in retail, a combination of both packs must be purchased. From the provided table, it can be observed that purchasing 4 packs of A and 1 pack of B costs IDR 10,400. Similarly, buying 3 packs of A and 2 packs of B results in a cost of IDR 9,800. Therefore, it can be concluded that buying 3 packs A and 2 packs B is the cheapest option.
Researcher: How is the quantity for purchasing pack A and pack B determined?
H1 : Identify the cheapest price by looking at the table.

Regarding the aspect of generalizing, prospective teachers from Groups 1 and 2, though a minority, demonstrated the ability to analyze mathematical situations, draw comparisons, and infer rules based on patterns and relationships. However, they struggled to establish their own set of regulations for analyzing mathematical scenarios, making analogies, and generalizing problems from the given data. In contrast, prospective teachers from Group 3 effectively generalized the problem by formulating equations involving variables and inequality symbols to represent the minimum number of cups that must be purchased.

In the third aspect, justifying, Group 3 evaluated the cost per cup in each pack, as depicted in Figure 2. Upon conducting the necessary calculations, it was determined that the price per cup was lower in pack A than in pack B. As a result, the conjecture favored purchasing pack A over pack B. This conjecture was developed through a comparative analysis, where the ratio for pack A was found to be more favorable than that of pack B. The possible ratios were identified as 4:1 or 3:2. Following the identification of the ratio, the price per cup was substituted into the equation to obtain the correct solution. In justifying their conclusions, Groups 1 and 3 provided the same rationale as the one used in formulating their conjectures, asserting that "the earlier conjecture provides the requested answer, namely the minimum price."


c. Ada, dengan memulai mengetahui lebih minimum mana biaya pack A dan Pack B.
Pack A = 2.200/12 = 183,3 sedangkan Pack B = 1.600/8 = 200.
Maka pasti lebih hemat Pack A, selanjutnya tinggal merasiokan perbandingan
Pack A harus lebih banyak, maka ada 2 opsi, yaitu : a.) 4 : 1 dan b.) 3 : 2
jika opsi a. total cup adl 56 dengan harga 10.400. Sedangkan opsi b. total cup
32 dan harganya 9.800. Maka sudah pasti lebih hemat opsi b.

Translate:
Yes, after comparing the cost per piece between Pack A and Pack B, it is evident that Pack A is more economical, with a cost of IDR183.3 per piece, and Pack B costs IDR 200. Considering the ratio of quantities between Pack A and Pack B, there are two options available: (a) 4:1 or (b) 3:2. If we choose option (a), a total of 56 cups can be purchased for a price of IDR 10,400, while option (b) gives us 32 cups at a price of IDR 9,800. Therefore, it is clear that option (b) is the cheaper option.

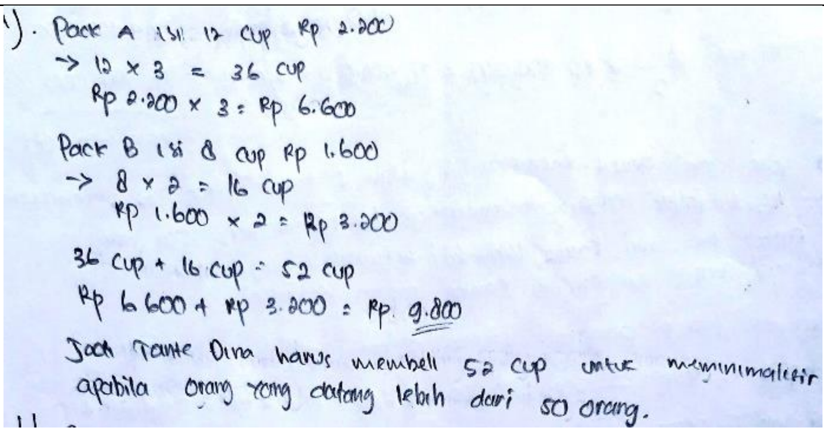
Figure 2. Participants' answers in Group 1 with high category

When asked to verify the accuracy of a conjectured solution without the availability of alternative answers, proficient educators demonstrate the ability to derive their own principles from the provided data. This can be achieved by comparing the price per cup between the two packs, which are 183.3 and 200, respectively, and identifying the ratio that corresponds to the most reasonable price. Subsequently, the educator confirmed the accuracy of the solution.

Prospective Teachers with Moderate Mathematical Reasoning

Prospective teachers in Groups 1 and 2, classified in the moderate category for mathematical reasoning, were generally able to identify specific rules for making predictions and processing solutions based on the available data (see Figure 3). They determined that purchasing 3 packs of type A and 2 packs of type B would provide the optimal solution. This calculation was based on the fact that pack A contains 12 cups while pack B contains 8 cups. To obtain a total close to 50 cups, 3 packs of A were purchased, with the remaining 2 cups sourced from pack B.

Conversely, the prospective teacher in Group 3 hypothesized that pack A had a multiple of 12 cups, while pack B had a multiple of 8 cups, resulting in a total slightly exceeding 50 cups. Specifically, purchasing 8 packs of B would yield 56 cups, which was more than the required 50 cups, leaving 6 cups surplus. Although this approach was initially considered correct, the prospective teacher ultimately provided the correct solution at a later stage: Dina should purchase 3 packs of A and 2 packs of B to minimize costs.

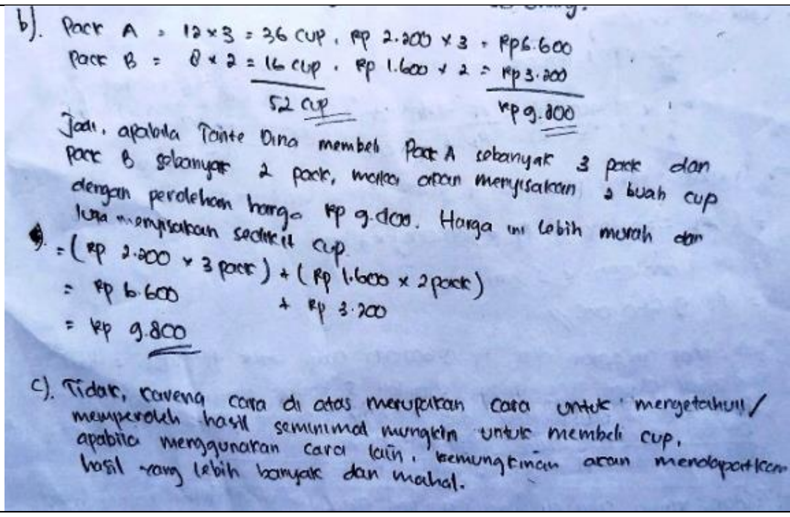


Translate:

Package A contains 12 cup Rp 2.200
-> 12 x 3 = 36 cup
Rp 2.200 x 3 = Rp 6.600

Package B contains 8 cup Rp 1.600
-> 8 x 2 = 16 cup
Rp 1.600 x 2 = Rp 3.200
36 cup + 16 cup = 52 cup
Rp 6.600 + Rp 3.200 = Rp 9.800\

So, Aunt Dina has to buy 52 cups to anticipate if more than 50 people attend.


b). Pack A = 12 x 3 = 36 cup, Rp 2.200 x 3 = Rp6.600
Pack B = 8 x 2 = 16 cup, Rp 1.600 x 2 = Rp3.200
52 cup Rp9.800
Jadi, apabila Tante Dina membeli Pack A sebanyak 3 pack dan Pack B sebanyak 2 pack, maka akan menyisakan 2 buah cup dengan perolehan harga Rp 9.800. Harga ini lebih murah dan juga menyisakan sedikit cup.
= (Rp 2.200 x 3 pack) + (Rp 1.600 x 2 pack)
= Rp 6.600 + Rp 3.200
= Rp 9.800
c). Tidak, karena cara di atas merupakan cara untuk mengetahui/ memperoleh hasil seminimal mungkin untuk membeli cup, apabila menggunakan cara lain, kemungkinan akan mendapatkan hasil yang lebih banyak dan mahal.

Translate:

Package A:
12 x 3 = 36 cups.
Rp 2.200 x 3 = Rp 6.600
Package B:
8 x 2 = 16 cups.
Rp 1.600 x 2 = Rp 3.200

Total cups: 36+16 = 52 cups
Total price: Rp. 6.600 + Rp. 3.200 = Rp 9.800

So, if Aunt Dina buys 3 packages of Package A and 2 packages of Package B, it will result in 52 cups with a total cost of Rp 9.800. This price is cheaper and also leaves less cup, which is:
-> (Rp. 2.200 x 3 packages) + (Rp. 1.600 x 2 packages)
-> Rp 6.600 + Rp 3.200
-> Rp 9.800

c). No, because the above method is intended to optimize/ minimize the cost of purchasing cups. If using other methods, the result may be more but also more expensive.

Figure 3. Participants' answers of conjecturing (above) and generalizing and justifying (below) in Group 1 with moderate category

In generalizing, prospective teachers in Groups 1 and 2 with moderate category summarized the main points from the conjecturing section. The interview with the moderate category of participants in Group 1 (M1) is provided in Transcript 2.

Transcript 2:

Researcher: Try to explain this one (pointing to the written answer)

M1 : The task requires the calculation of the minimum capital from 50 people. Multiplying 12 by 3 equals 36. The price per pack is 2200 multiplied by 3 to equal 6600. Pack A, which contains 12 cups is chosen. Then, one cup from pack B, which costs 1600 and contains 8 cups, is added. Therefore, if we multiply 8 cups by 2, we get 16 cups. Similarly, if we double the price of 1600, we get 3200. The total amounts to 52, adding up 36 and 22, with a price of 6600 plus 3200, which equals 9800.

Researcher : What were your reasons for choosing 3 as the quantity for pack A? Have you *considered testing other numbers such as 4 or 5?*

M1 : Perhaps so that the number isn't excessively large; for example, if 12 times 4 = 48, then adding from pack B gets 48+8. This would result in a large number of cups (i.e. 6 cups). On the other hand, if I decide to buy three packs of pack A, that means 3 times 12 equals 36 cups. So, I opted for this solution to avoid excessive purchase of cups, limiting it to only two cups in total, calculated by adding three sets of twelve ounces and two sets of eight ounces.

Researcher : Have you tried other number combinations for both packs?

M1 : I have, but the result leaves a residual difference of more than 2, so I didn't write it down here.

Researcher : How can the solution to this problem be generalized?

M1 : The generalization is calculated by multiplying the price of the first pack by the number of packs purchased and adding it to the product of the price of the second pack (pack B) multiplied by the number of packs purchased. This is my proposed generalization.

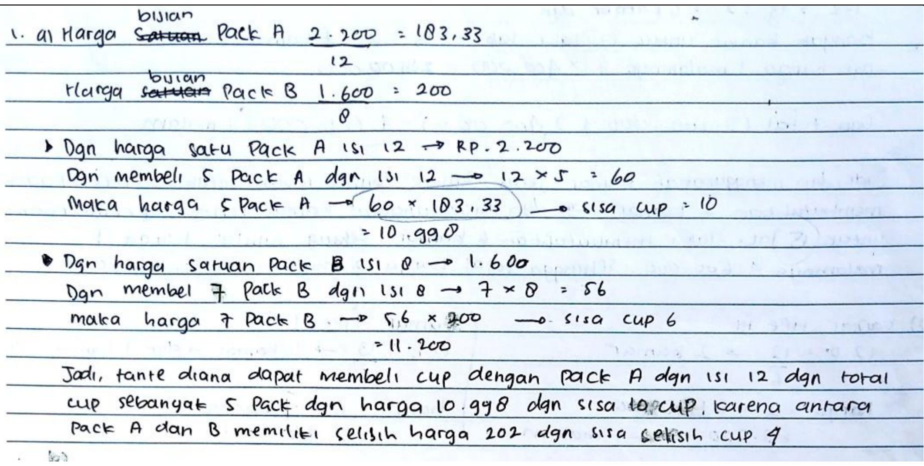
Researcher:Is there an alternative method of purchasing the cups that has not been presented?

M1 : No, because the answer above is a way to get the minimum possible price to buy cups from pack A and pack B. Using different methods may result in more expensive or leftover cups.

Transcript 2 details that participants in the moderate category, during the justification phase, provided an explanation based on the information previously presented. In contrast, during the conjecturing phase, a prospective teacher from Group 3 opted for unmixed cups, selecting the least expensive option with the smallest remaining quantity. An alternative cost-minimizing method was proposed in the explanatory section. This approach involved purchasing 52 cups by acquiring 3 packs of A and 2 packs of B, at a total cost of IDR 9,800, with only 2 cups remaining unused. This strategy was found to be more economical than the alternative method of purchasing 5 packs of A for IDR 11,000.

Prospective Teachers with Low Mathematical Reasoning

The following section discusses the responses of prospective teachers classified in the low mathematical reasoning category on the test. When calculating the required number of packs of A and B, these teachers assumed that cups could only be purchased from one type of pack (either pack A or pack B, but not both), as shown in Figure 4. Consequently, they resorted to guesswork, selecting the multiple closest to 50 cups for each pack. Since pack A contains 12 cups, they chose to purchase 5 packs of A, yielding 60 cups. Similarly, to reach a total of 50 cups, they opted to buy 7 packs of B, each containing 8 cups, resulting in 56 cups. After these calculations, 10 cups from pack A and 6 cups from pack B remained unused. Regarding the cost, 5 packs of A amounted to IDR 11,000 (2200 x 5), while 7 packs of B cost IDR 11,200 (1600 x 7).


1. a) Harga ~~satuan~~ bijian Pack A 2.200/12 = 183,33
Harga ~~satuan~~ bijian Pack B 1.600/8 = 200
• Dgn harga satu Pack A isi 12 → Rp. 2.200
Dgn membeli 5 Pack A dgn isi 12 → 12 × 5 = 60
Maka harga 5 Pack A → 60 × 183,33 → sisa cup = 10
= 10.998
• Dgn harga satuan Pack B isi 8 → 1.600
Dgn membeli 7 Pack B dgn isi 8 → 7 × 8 = 56
maka harga 7 Pack B → 56 × 200 → sisa cup 6
= 11.200
Jadi, tante diana dapat membeli cup dengan Pack A dgn isi 12 dgn total
cup sebanyak 5 Pack dgn harga 10.998 dgn sisa ~~10~~ cup, karena antara
Pack A dan B memiliki selisih harga 202 dgn sisa selisih cup 4

Translate:

1. a) Price per Pack A = 2.200/12 = 183,33
b) Price per Pack B = 1.600/8 = 200

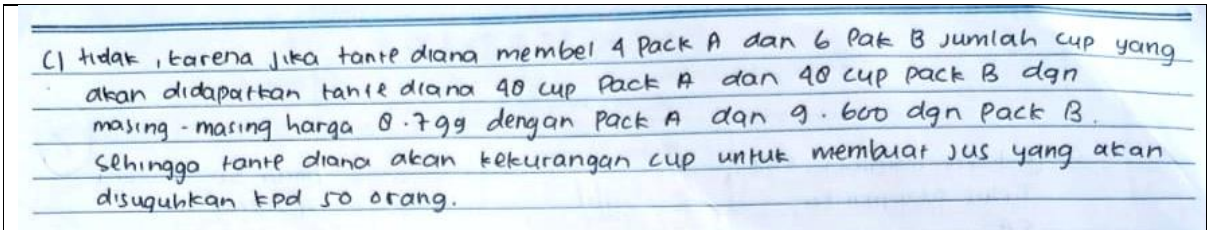
- With the unit price of Pack A, which contains 12 = Rp 2.200
By buying 5 Pack A with 12 contents = 12 x 5 = 60
Then the price of 5 Pack A = 60 x 183,33 = 10.998 with 10 cups remaining

- With the unit price of Pack B, which contains 8 = 1.600
By buying 7 Pack B with 8 contents = 7 x 8 = 56
Then the price of 7 Pack B = 56 x 200 = 11.200 with 6 cups remaining

So, Aunt Diana can buy cups with Pack A containing 12 with a total of 5 packs for 10.998 with 10 cups remaining, because between Pack A and Pack B, there is a price difference of 202 with a difference of 4 remaining cups.

Figure 4. Participants' answers of conjecturing aspect in Group 3 with low category

In the aspect of generalization, none of the three groups incorporated equations into their responses. However, this aspect was characterized by a reiteration of the content presented in the conjecturing section. In the third aspect, justification, both Groups 1 and 3 did not provide adequate reasoning for their answers. In contrast, Group 3 offered a justification in the form of a descriptive explanation of their solution, as illustrated in Figure 5.


c) tidak, karena jika tante diana membel 4 Pack A dan 6 Pak B jumlah cup yang
akan didapatkan tante diana 40 cup Pack A dan 48 cup pack B dgn
masing-masing harga 8.799 dengan Pack A dan 9.600 dgn Pack B.
sehinggo tante diana akan kekurangan cup untuk membuat jus yang akan
disuguhkan kpd 50 orang.

Translate:
No, because if Aunt Dina purchases 4 Pack A and 6 Pack B, she will only receive 48 cups from each pack, leaving a shortage of cups for serving juice to 50 people. The price of Pack A is 8,799 and Pack B is 9600.

Figure 5. Participants' answers of justifying aspect in Group 3 with low category

This participant provided a justification by presenting a counterexample, distinct from the one outlined in the conjecturing aspect. The participant's reasoning, categorized at this level, was based on the assumption that cup purchases could only be made from a single pack type (either pack A or pack B), which led to the exclusion of the possibility of finding a combination of multiples of 12 and 8 that, when summed, would exceed 50 but remain relatively close to it.

Differences in Mathematical Reasoning Across Prospective Teacher Groups

The participants' test scores were categorized into grades and corresponding proficiency levels based on the guidelines presented in Table 1. Results from the homogeneity test indicated that the variance between groups was not homogeneous ($p = 0.020$), suggesting significant differences in the variability of mathematical reasoning scores across the groups. This lack of homogeneity could be attributed to various factors, including differences in the participants' understanding of the material, the teaching methods employed, or variations in the difficulty level of numeracy questions administered across semesters. Given that the assumption of homogeneity was not satisfied, the application of non-parametric tests, specifically the Shapiro-Wilk test, was deemed appropriate for subsequent analysis.

The Shapiro-Wilk test was used to assess whether the mathematical reasoning data followed a normal distribution. The test results indicated that the data were not normally distributed ($p = 0.003$), suggesting that the distribution of mathematical reasoning scores deviates from a normal distribution, potentially due to the presence of outliers or skewed data. This deviation may also point to substantial variability in performance, with some groups exhibiting a broad range of very high or very low scores. Consequently, non-parametric analyses, such as the Kruskal-Wallis test, were considered more suitable for further analysis under these conditions.

Table 2. Descriptive statistics by groups

Group (G)	Mean (M)	Standard Deviation (SD)	Standard Error (SE)	Coefficient of Variation
Group 1	40.833	17.158	2.112	0.420
Group 2	42.724	14.424	1.775	0.345
Group 3	52.879	22.129	2.724	0.418

As indicated in Table 2, the mean score for Group 3 is the highest ($M = 52.879$, $SD = 22.129$), suggesting that prospective mathematics teachers in this group generally possess stronger mathematical reasoning skills compared to those in the other groups. The higher standard deviation for Group 3 also reflects greater variability in performance, which may point to differences in individual mastery of the material or reasoning abilities. In contrast, Group 1 demonstrated a lower mean ($M = 40.833$, $SD = 17.158$), which may suggest a weaker grasp of foundational numeracy concepts among early-semester students. Notably, 38 out of 66 students (approximately 57.58%) in Group 1 scored above the overall mean, indicating that a majority of students in this group exhibited relatively strong mathematical reasoning skills.

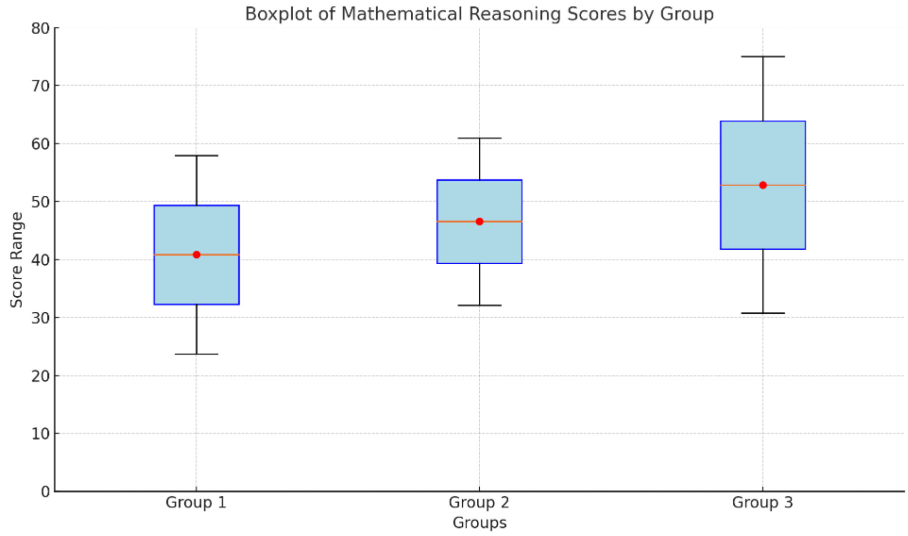


Figure 6. Mathematical reasoning scores by groups

Figure 6 illustrates the range of mathematical reasoning scores (minimum, median, and maximum) for each group, emphasizing the variability in performance as well as the central tendencies. Group 3 exhibits the highest median score, with greater variability in comparison to Groups 1 and 2.

Table 3. The Kruskal-Wallis test depends on the groups

Factor	Statistic	df	p
Group	12.544	2	0.002

The results of the Kruskal-Wallis test, as shown in Table 3, indicate a significant difference in mathematical reasoning abilities between the three groups of prospective mathematics teachers (H(2) = 12.544, p = 0.002). This suggests that at least one group exhibits significantly different mathematical reasoning skills compared to the others.

Table 4. Post hoc test results across groups

Comparison	z	Wi	Wj	p	pbonf	pholm
1 - 3	-0.231	88.174	90.477	0.817	1.000	0.817
1 - 5	-3.176	88.174	119.848	0.001	0.004	0.004
3 - 5	-2.945	90.477	119.848	0.003	0.010	0.006

Further examination of the Post Hoc test results, presented in Table 4, provides insight into which specific groups demonstrated significant differences. There was no significant difference in mathematical reasoning abilities between Groups 1 and 2 (p = 0.817), indicating their performance levels were relatively similar. However, significant differences were observed between Group 1 and Group 3 (p = 0.001), as well as between Group 2 and Group 3 (p = 0.006). These findings suggest that Group 3, consisting of students in the higher semester, displayed markedly better mathematical reasoning abilities than the other two groups. This may reflect the positive impact of additional learning experience and the increased complexity of the material encountered in the higher semester on the development of mathematical reasoning skills.

Differences in Prospective Teachers' Mathematical Reasoning Aspects (Conjecturing, Generalizing, and Justifying) Across Groups

This study focuses on the three aspects of mathematical reasoning—conjecturing, generalizing, and justifying—which were analyzed based on specific indicators. The following presents the detailed results of the quantitative tests for these aspects.

Conjecturing

The homogeneity test for the conjecturing aspect revealed that the data across the groups were homogeneous ($p = 0.601$), suggesting that the variance in conjecturing ability was similar among the groups. This indicates that the variability in each group's ability to conjecture is not significantly different. However, while homogeneity was confirmed, the normality test indicated that the data did not follow a normal distribution, highlighting the need for non-parametric analysis for more accurate conclusions.

Despite the homogeneity of the conjecturing data, the Kruskal-Wallis test results showed no significant difference in conjecturing ability among the three groups ($p = 0.094$). This suggests that the ability to make mathematical conjectures is not significantly influenced by the semester or level of education, and may be more dependent on individual ability or other factors not directly related to the educational level.

Generalizing

Regarding the generalizing aspect, it was found that the data from the three groups were not homogeneous ($p < 0.001$). The Shapiro-Wilk test further indicated that the data did not meet the assumption of normality ($p < 0.01$). Consequently, non-parametric analysis, specifically the Kruskal-Wallis test, was employed.

The Kruskal-Wallis test revealed no significant difference between the groups ($p = 0.061$), although the p-value was close to the threshold for significance. This result may suggest that the ability to generalize, or make generalizations from specific cases, develops over time but does not show substantial differences between semesters. It is possible that this ability takes longer to develop or is not adequately fostered by the curriculum in each semester.

Justifying

For the justifying aspect, the data from the three groups were not homogeneous ($p = 0.010$; $p < 0.05$) and did not meet the assumption of normality ($p < 0.001$), necessitating the use of non-parametric analysis through the Kruskal-Wallis test. The results of the Kruskal-Wallis test for the justifying aspect showed a significant difference ($p < 0.001$), indicating that the ability to justify, or provide reasons for an answer, significantly differed across groups. This difference may reflect improvements in analytical and critical thinking skills acquired through experience and education.

Table 5. Post hoc test of mathematical reasoning aspects

Comparison	z	Wi	Wj	p-value	p_bonf	p_holm
1 - 2	-1.822	78.091	96.212	0.068	0.205	0.068
1 - 3	-4.636	78.091	124.197	< .001	< .001	< .001
2 - 3	-2.814	96.212	124.197	0.005	0.015	0.010

The post hoc analysis, presented in Table 5, revealed no significant difference between Groups 1 and 2 ($p = 0.068$) for the justifying aspect. However, significant differences were observed between Group 1 and Group 3 ($p < 0.001$), as well as between Group 2 and Group 3 ($p = 0.005$). These findings suggest that the ability to justify answers improves significantly in higher semesters, likely due to the increased complexity of the material taught and the additional experience in analyzing and explaining mathematical concepts.

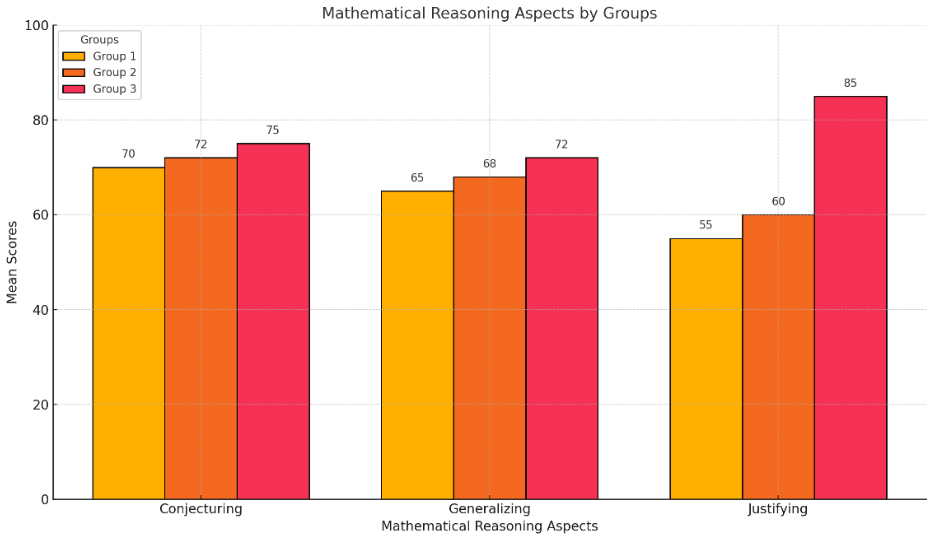


Figure 7. Mathematical reasoning aspects by groups

The quantitative analysis presented above indicates differences in mathematical reasoning across the three groups of prospective teachers (see Figure 7). Notably, the most pronounced difference was observed in the aspect of justifying.

Discussion

The findings of this study indicate that the mathematical reasoning abilities of prospective teachers vary across different aspects. While the majority demonstrated proficiency in formulating conjectures, they encountered substantial difficulties in generalizing and justifying mathematical ideas. The process of conjecturing was predominantly driven by inductive reasoning, supported by empirical observations that served to validate mathematical statements (Knuth et al., 2019). Nevertheless, exclusive reliance on empirical evidence does not suffice for rigorous mathematical reasoning, as it lacks the deductive strength required to construct logically sound arguments that generalize from particular cases to broader mathematical principles (Yopp, 2015; Zaslavsky et al., 2012).

In solving estimation tasks, prospective teachers generally employed their most accessible strategy—namely, the use of standard algorithms. Some represented their solutions through algebraic expressions involving two variables, while others utilized tabular representations. This variation reflects differing levels of mathematical sophistication, as learners who are capable of integrating relevant information and mathematical concepts tend to generate diverse and meaningful solution strategies (Marsitin et al., 2022; Syarifuddin et al., 2020). In more advanced cases, prospective teachers substituted variables with appropriate values, identified common factors (such as those of 12 and 8), summed the values, and approximated the result to a target value (e.g., 50). Such strategies were typically employed by individuals with medium to high proficiency, who exhibited fluency in reasoning processes. Conversely, less proficient participants often relied on trial-and-error methods to approach the correct solution. Although these individuals acknowledged that the algorithmic approach was the most efficient for reaching the correct answer, such methods tend to lack the conceptual depth required for a comprehensive understanding of the underlying mathematical structures, particularly in contexts involving linear programming optimization.

The variability in reasoning approaches observed among participants may be attributed to differences in prior knowledge and experiential background. Those with stronger foundational understanding and greater exposure to mathematical problem solving were more likely to formulate creative and insightful conjectures. It is demonstrating an ability to draw upon previously acquired concepts and experiences to inform their reasoning (Lathifaturrahmah et al., 2023).

Difficulties in generalizing were prominently observed among prospective teachers, many of whom approached the task through a procedural, linear lens. This approach typically involved a sequence of fixed steps, such as determining the number of cups needed, calculating the cost per cup, and comparing total prices—procedures that were often initially explored through trial-and-error methods during the conjecturing phase. Furthermore, generalization entails identifying similarities and differences related to a mathematical object or relationship (Jeannotte & Kieran, 2017). However, the findings reveal that while participants were capable of recognizing patterns and relationships, they lacked the ability to abstract these observations into general rules or principles for analyzing mathematical situations, drawing analogies, and constructing broader generalizations. This aligns with the findings of Rodrigues et al. (2021), who highlighted the prevalent struggles among students, teachers, and prospective teachers in executing generalization, despite a fundamental understanding of the problem at hand. These challenges often stem from a tendency to focus narrowly on isolated pieces of information, rather than synthesizing multiple elements to generate a generalized conclusion (Parameswari et al., 2023). The core difficulty lies in extending reasoning beyond immediate contexts, a process that is critical for applying mathematical thinking to novel scenarios (Lannin et al., 2011; Herbert et al., 2015).

With respect to the justification component of reasoning, many prospective teachers demonstrated limited capacity to move beyond their initial conjectures. Although they were able to utilize the provided data, they encountered difficulties in constructing logically valid arguments grounded in formal mathematical principles. This issue was particularly notable among participants in Groups 1 and 3, where justification often amounted to a mere restatement of previously determined answers without further elaboration or evidence. While some participants attempted deductive reasoning—drawing specific conclusions from general premises—this reasoning was frequently superficial and lacked rigor. These findings are consistent with those of Hidayah et al. (2020; 2023), who observed that prospective teachers tend to verify conjectures through deductive means, albeit with limited depth. Stylianides (2009) further emphasized that persistent misconceptions about the nature of mathematical proof—particularly the role of empirical evidence—continue to impede prospective teachers' understanding, even after formal instruction in proof techniques. Supporting this, Rodrigues et al. (2021) noted that students often fail to recognize that empirical examples are inadequate for justification, leading them to erroneously regard statements as universally valid, despite the presence of counterexamples or exceptions (Lannin et al., 2011).

The difficulties encountered in constructing justifications are often exacerbated by limited conceptual understanding and insufficient exposure to analogous problem types (Kristayulita et al., 2020). As highlighted by Lo and McCrory (2012), it is essential for prospective teachers to cultivate their justificatory abilities at three progressive levels: executing formal proofs, comprehending the nature and function of proofs, and adapting proof-related ideas to suit varying developmental stages of learners. The present study reinforces the findings of Lesseig (2016), who reported that prospective teachers frequently possess a fragile grasp of the justification process. Effective mathematical justification necessitates the articulation of a coherent argument grounded in established mathematical principles and prior knowledge (Lannin et al., 2011; Stylianides, 2007). While counterexamples are instrumental in refuting invalid claims, the practice of verifying a general statement through specific examples alone is insufficient to establish its universal validity (Lesseig, 2016). Nevertheless, examples remain valuable tools in the formation of conjectures and in probing the boundaries of generalizations (Lannin et al., 2006; Pedemonte & Buchbinder, 2011).

The findings also reveal that prospective teachers with more advanced academic experience—particularly those in their fifth semester and classified under Group 3—demonstrated stronger mathematical reasoning in the area of justification than their peers in earlier semesters. This suggests a positive correlation between increased mathematical exposure and reasoning proficiency, as previously indicated by Ikram et al. (2020). However, prior research by Bergqvist et al. (2008) and Lithner (2000) has shown that undergraduate students often rely on procedural knowledge and familiar algorithms rather than engaging deeply with the conceptual foundations of mathematics. This underscores the necessity for further investigation into how undergraduate coursework influences the development of mathematical reasoning and content knowledge.

In light of these findings, it is imperative that teacher education programs emphasize tasks that foster the development of mathematical reasoning, particularly in the domains of generalization and justification. As Stylianides and Stylianides (2006) argue, cultivating these competencies is vital for preparing future educators to navigate the complexities of mathematics instruction. Moreover, the ability to justify and generalize mathematical ideas is foundational to effective mathematics teaching (Herbert et al., 2015). To that end, sustained professional development opportunities and instructional environments that promote the application of mathematics to authentic, real-world contexts are essential for enhancing prospective teachers' reasoning skills and equipping them for the demands of their future professional roles (Hiebert et al., 2003).

CONCLUSION

This study offers critical insights into the mathematical reasoning abilities of prospective mathematics teachers by identifying significant variations in their reasoning performance across different stages of their academic progression. The results revealed that while conjecturing—often supported by inductive reasoning and empirical validation—was generally well developed among participants, substantial difficulties emerged in the areas of generalization and justification. These findings suggest that although algorithmic fluency may support initial conjectures, it does not necessarily translate into the capacity to formulate generalized conclusions or construct logically sound justifications. Furthermore, participants in more advanced semesters demonstrated enhanced reasoning capabilities, particularly in justification, which can be attributed to their increased exposure to abstract mathematical concepts and cumulative learning experiences throughout the teacher education curriculum. Nonetheless, the observed dependency on procedural approaches and the limited success in developing generalized and deductively justified solutions underscore a critical gap in the cultivation of higher-order mathematical thinking within current pre-service teacher education programs.

Despite its contributions, this study is constrained by several limitations. The research focused on a single cohort within a specific institutional context, which may restrict the broader applicability of its conclusions to diverse educational settings. Moreover, the emphasis on numeracy-based tasks may not encompass the full range of reasoning proficiencies required in other mathematical domains such as algebra, geometry, or calculus. These constraints highlight the need for future research that systematically investigates mathematical reasoning across varied content areas and instructional environments. Longitudinal studies could provide richer insights into the developmental trajectory of reasoning competencies throughout the duration of teacher preparation. Additionally, experimental studies integrating targeted instructional interventions—particularly those aimed at enhancing generalization and justification—are recommended to strengthen the conceptual depth and pedagogical readiness of prospective mathematics teachers. Such initiatives are essential for fostering robust mathematical reasoning skills, thereby contributing to the advancement of mathematics education quality and effectiveness.

ACKNOWLEDGMENTS

The authors extend their sincere appreciation to the prospective teachers who willingly participated as respondents in this study, thereby contributing significantly to the smooth and effective implementation of the research.

DECLARATIONS

Author Contribution	: DH: Conceptualization, Data Curation, Investigation, Formal Analysis, Methodology, Writing–Original Draft Preparation, and Writing–Editing. S: Conceptualization, Methodology, Writing–Review & Editing, and Validation. HS: Conceptualization, Methodology, and Writing–Review & Editing. ZAH: Data Curation, Visualization, Writing–Review & Editing, and Formal Analysis. SO: Methodology and Validation.
Funding Statement	: This research was conducted without any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
Conflict of Interest	: The authors declare no conflict of interest.
Additional Information	: Additional information is available for this paper.

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